PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH

PROCEEDINGS

OF

THE ROYAL SOCIETY OF EDINBURGH

VOL. LIX

EDINBURGH
PRINTED BY NEILL AND COMPANY, LIMITED
MCMXL

CONTENTS

*** The thanks of the Society are due to the Carnegie Trust for the Universities of Scotland for grants towards the costs of the illustrations, tables, etc., in these papers.**

// PROCEEDINGS

OF THE

ROYAL SOCIETY OF EDINBURGH.

VOL. LIX. 1938–39.

I.—The Molecular Spectra of the Hydrogen Isotopes. I.—Application of the Rotating Vibrator Model to the States of D_2.

By **Ian Sandeman,** D.Sc., Late Carnegie Research Scholar in the University of St Andrews.

(MS. received August 22, 1938. Read November 7, 1938.)

THE theory of the rotating vibrator has been developed by the late J. L. Dunham (1932). The essential step in Dunham's treatment of this question is his replacement of the potential expression occurring in the Schrödinger equation for the diatomic molecule by an arbitrary function in terms of the nuclear separation. When this replacement is made, the Schrödinger equation can be solved by methods developed by Wentzel (1926), Brillouin (1926), and Kramers (1926), and the energy of the rotating vibrator can be expressed as a power series in the quantum numbers in a form convenient for application to spectral data.

Dunham's treatment of this problem is his principal contribution to mathematical spectroscopy. The initial step of replacing the molecular potential by an arbitrary function, however, raises some difficulties, and Dunham's work has not been fully accepted by spectroscopic workers.

The justification for the replacement of the molecular potential by an arbitrary function in terms of the nuclear separation rests on practical grounds. For molecules like that of neutral hydrogen which contain two singly charged nuclei (A, B) and two electrons (1, 2) the molecular potential is:

$$\frac{e^2}{r_{AB}}-\frac{e^2}{r_{A1}}-\frac{e^2}{r_{B1}}-\frac{e^2}{r_{A2}}-\frac{e^2}{r_{B2}}+\frac{e^2}{r_{12}}. \qquad (1)$$

Such expressions are easy to write down, but lead to difficulty, when an attempt is made to solve the Schrödinger equation. (There has been in

recent years a large number of investigations on this question, notably those of Heitler and London, Sugiura, Hylleraas, Wang, Rosen, Weinbaum, and James and Coolidge. See a paper by Van Vleck and Sherman (1935) in which a convenient summary is given.)

In applying Dunham's theory of the rotating vibrator we cut the Gordian knot by replacing the molecular potential by a general potential expansion which Dunham writes as:

$$U = a_0\xi^2(1 + a_1\xi + a_2\xi^2 + a_3\xi^3 + \ldots). \qquad (2)$$

Here ξ expresses the ratio of $(r - r_e)$, the deviation of the internuclear distance from its equilibrium value, to r_e the equilibrium value, or $\xi = (r - r_e)/r_e$; a_1, a_2, a_3, . . . may be called "configuration constants," since they differ for different states and so bear some relation to the configuration of the molecule; and a_0 bears the relation to the usual constants of spectroscopic analysis $a_0 = \omega_e^2/4B_e$. (ω_e is the classical frequency of small oscillations in wave-numbers, and $B_e = h/8\pi^2\mu r_e^2 c$, μ being the reduced mass of the molecule.) Essentially we are replacing the molecule by a model in which the two nuclei are held at a given distance, r_e, by forces which have a potential depending only on the internuclear distance.

The assumption that this model will represent molecular behaviour is evidently a good one, since we know that for any given molecular state the molecule adjusts itself to an equilibrium internuclear distance which is always the same for the same state. The assumption has been fruitful in providing the basis of a mathematical theory for the analysis of the vibrational and rotational structure of spectra. The gain achieved must be compensated by a loss somewhere.

There is in fact a loss in two different ways. First, by the introduction of an arbitrary function we have abandoned any claim to specify the structure of the function U or to relate the U of one state to that of another. We know from the work of Hund that couplings occur between the various intra-molecular motions which have no counterpart in our simplified model. By using an arbitrary function we are therefore adopting a procedure which tends to lose sight of a whole series of well-established molecular phenomena. Secondly, we have left the solid ground of fact behind in so far as we cannot be sure that our arbitrary function will be the actual potential function of the molecule. We know that we may have to replace the constant of integration K(K + 1), which corresponds to the rotation quantum number of older theory, by a more complicated expression. This would have the effect, as Davidson (1932) has shown, of altering the configuration constants of equation (2) by

factors of the form $1+cu_e^2$, where c is a constant and u_e a contraction for $2B_e/\omega_e$. (Davidson uses the Greek letter κ instead of u_e.) Again, we have to contend with the fact that uncoupling may occur with increase in the speed of rotation and with ambiguities introduced by rotational doubling. We may also meet with perturbations which find no place in the simplified model and may invalidate our analysis.

In spite of these difficulties there is a field within which Dunham's theory can be usefully applied. The difficulty introduced by the form of the constant of integration and by rotational doubling may be avoided by confining attention to Σ states. To avoid the complication of perturbations we may still further confine attention to the regular Σ states which lie outside the regions where there are known states with which they may interact. We can then test whether Dunham's analysis gives self-consistent results. If so, we can be sure that there has been no appreciable uncoupling.

The special simplicity of the hydrogen states arises from the fact that the interactions of the spins with the other motions is very weak (Richardson, 1934, pp. 45, 46). The states accordingly come under the simplest type to be considered (Hund's Case *b'*). As all the states that have been interpreted with certainty have one electron in the $1s$ state, the component of orbital angular momentum of the excited electron round the nuclear axis, λ, is the same as Λ for the molecule. When the molecule is revolving slowly, *i.e.* when it is in a state of low rotation quantum number, Λ interacts with the nuclear angular momentum N to form a resultant which is quantised with the quantum number K, and K takes values Λ, $\Lambda+1$, $\Lambda+2$, . . .

For the case $\Lambda=0$ we have $K=N$, and the angular momentum of the nuclei is itself quantised. We could imagine no case coming nearer that of the model. The regular Σ states of the hydrogen isotopes constitute an ideal field for the application of the rotating vibrator model. In the case of such states it is not very probable that the potential function found for the model differs from the true function for the molecule.

In a previous paper the writer (1935) has attempted an analysis of the $1s\sigma 2p\sigma\,{}^1\Sigma$ and $1s\sigma 2s\sigma\,{}^3\Sigma$ states of H_2. Data for the corresponding states of D_2 are now available from the work of Dieke and his collaborators, and these, owing to the greater mass of the D_2 molecule and the consequent smallness of the constant u_e, should form better material for the application of Dunham's theory.

Data for the corresponding states of HD are also available. These data are, however, not quite so satisfactory, and Dieke (1936) explains the reason. The spectrum of D_2 is, like that of H_2, best obtained from the

pure gas, whereas the spectrum of HD is best obtained with a mixture containing 50 per cent. of each gas. The result is that the HD plates contain the spectra of H_2 and D_2 as well as that of HD, and the overlapping of lines makes accurate measurement more difficult.

The practical task of applying Dunham's analysis is by no means an easy one. Following Dunham, we must first attempt to fit all the known term-differences into the term form:

$$F(v, K) = \sum_{lj} Y_{lj}(v+\tfrac{1}{2})^l K^j (K+1)^j, \qquad . \quad . \quad . \quad (3)$$

calculating as many of the coefficients Y_{lj} as we need to represent the experimental data accurately. Dunham gives theoretical expressions for all the Y_{lj} for which $l+j$ does not exceed 4, coefficients which do not involve any configuration constant beyond a_6. Once the Y_{lj} have been calculated, we can use Dunham's expressions for the first few of these to calculate the structure constants of the state.

The experimental data to which Dunham's analysis is to be applied are in the form of term-differences obtained by taking the difference between pairs of lines which have an upper state in common, and are therefore of the form:

$$F(v_a, K_a) - F(v_b, K_b). \qquad . \quad . \quad . \quad . \quad (4)$$

It is essential to arrange these term-differences in some systematic way for the purpose of study.

By selecting all those for which $v_a = v_b$ we obtain the rotational term-differences, which can be arranged in the usual way in a scheme according to the value of the vibration quantum number and the pair of values K_a and K_b which, owing to the operation of the selection rules, differ by 2. The scheme of rotational term-differences consequently takes the following form:—

ΔK.	$v=0$.	1.	2.	. . .	
2–0	$F(0, 2) - F(0, 0)$	$F(1, 2) - F(1, 0)$	$F(2, 2) - F(2, 0)$	. . .	
3–1	$F(0, 3) - F(0, 1)$	$F(1, 3) - F(1, 1)$	$F(2, 3) - F(2, 1)$	. . .	(5)
4–2	$F(0, 4) - F(0, 2)$	$F(1, 4) - F(1, 2)$	$F(2, 4) - F(2, 2)$	. . .	
. . .	. . .	. . .	. . .	. . .	

Similarly we may pick out all the term-differences which have $K_a = K_b$, and obtain a scheme of term-differences which may be called the vibrational term-differences. The quantum number v is not subject to rigid selection rules, and we may obtain a large number of such differences. It is, however, convenient to restrict attention to those for which v_a and v_b differ by unity, as these usually provide quite sufficient material for study.

Our scheme of vibrational term-differences then takes the following form:—

Δv.	K = 0.	1.	2.	. . .	
1–0	$F(1, 0) - F(0, 0)$	$F(1, 1) - F(0, 1)$	$F(1, 2) - F(0, 2)$	. . .	
2–1	$F(2, 0) - F(1, 0)$	$F(2, 1) - F(1, 1)$	$F(2, 2) - F(1, 2)$	. . .	(6)
3–2	$F(3, 0) - F(2, 0)$	$F(3, 1) - F(2, 1)$	$F(3, 2) - F(2, 2)$	. . .	
. . .	. . .	. . .	. . .	. . .	

As Dieke (1936) has pointed out, the two sets of term-differences are not independent. The difference between any two adjacent members of the rotational set on the same horizontal line is the same as the difference between the appropriate pair of vibrational term-differences which are separated from one another by one intervening column. For example we have:

$$\{F(1, 2) - F(1, 0)\} - \{F(2, 2) - F(2, 0)\} \equiv \{F(2, 0) - F(1, 0)\} - \{F(2, 2) - F(1, 2)\}. \quad (7)$$

This fact is of value in testing whether the experimentally determined rotational and vibrational term-differences are consistent with one another.

It is evident that from the vibrational term-differences we can calculate all the Y_{lj} except those for which $l=0$—the pure-rotation coefficients—while from the rotational term-differences we can calculate all the Y_{lj} except those for which $j=0$—the pure-vibration coefficients. The remaining Y_{lj}—the "reciprocal-action" coefficients of the older terminology—can be found from either set of term-differences.

The writer's method of carrying out the analysis of a state is, whenever possible, to calculate the pure-vibration and reciprocal-action coefficients from the vibrational term-differences, and then to obtain the pure-rotation coefficients by substituting the reciprocal-action coefficients in the rotational term-differences. This procedure is justified by the fact that very accurate values of the vibrational term-differences can generally be obtained by averaging these over a large number of data.

The fact that most spectroscopic workers tabulate only the rotational term-differences and apply the usual rotational analysis to their bands has not tended to bring out the importance of the vibrational term-differences. If we are to carry out a thoroughgoing analysis of any state on a basis of Dunham's work, the pure-vibration coefficients and the reciprocal-action coefficients for which $j=1$ are important. The pure-rotation coefficients, except Y_{01}, are relatively less important, Y_{02} fulfilling the function of checking the self-consistency of the analysis.

Once the vibrational term-differences for any state have been extracted and arranged as in scheme (5), they are usually found to form an uneven

array, extending to higher values of K only when Δv is low and *vice versa*, with gaps and doubtful members round the edges of the table. It is accordingly generally necessary to rule off a portion of the table as trustworthy for the calculation of the Y_{ij}. Table I shows the vibrational

TABLE I.—VIBRATIONAL TERM-DIFFERENCES OF THE $1s\sigma 2p\sigma\,{}'\Sigma$ STATE OF D_2.

Δv.	K = 0.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
1–0	942·31 −0·02	941·61 0·00	940·18 0·01	938·06 0·02	935·23 0·00	931·79 0·00	927·75 0·00	923·21 0·04	918·14 0·04	912·71 0·08	906·87* 0·05
2–1	922·93 0·03	922·24 −0·02	920·97 −0·04	919·12 −0·02	916·66 −0·03	913·62 −0·05	910·12 0·00	906·07 −0·02	901·58* −0·04	896·84* 0·06	
3–2	904·66 0·05	904·07 0·03	902·97 0·05	901·26 0·02	899·09 0·06	896·35 0·03	893·13 0·02	889·53* 0·07			
4–3	887·05 −0·02	886·55 0·00	885·51 −0·01	883·98 −0·01	881·95 −0·01	879·45 −0·01	876·51 0·00	872·97 −0·15			
5–4	869·99* 0·00	869·51 0·00	868·54 −0·02	867·13 0·00	865·21 −0·03	862·89 −0·02	860·10* −0·04				
6–5		852·74 −0·01	851·86 0·01	850·51 0·00	848·75 0·01	846·56 0·03	843·96* 0·05				
7–6		836·20 0·01	835·32 −0·02	834·09 0·02	832·40 0·01	830·29 −0·01	827·77* −0·03	824·97 0·07			
8–7		819·87 0·00	819·06 0·00	817·86 0·00	816·23 −0·02	814·25 0·01	811·85* 0·01	809·20* 0·14			
9–8		803·68 −0·27	802·88 −0·28	801·70 −0·29	800·26* −0·17	798·27* −0·21					

term-differences for the $1s\sigma 2p\sigma\,{}'\Sigma$ state of D_2 obtained by averaging all the available data from the $2s\,{}'\Sigma \rightarrow 2p\,{}'\Sigma$ band system measured by Dieke (1936), and the $3d\,{}'\Sigma \rightarrow 2p\,{}'\Sigma$ and $3d\,{}'\Pi \rightarrow 2p\,{}'\Sigma$ band systems measured by Dieke and Lewis (1937). The experimental values in the table differ widely in trustworthiness. Most of the values towards the top left-hand corner of the table have been found by averaging a dozen or more term-differences depending on lines free from coincidences with lines of other known systems. Round the edge of the table the observed values are of less certain accuracy, and a few depend on rather doubtful band lines. Term-differences derived from only one pair of lines are marked with an asterisk.

The figures underneath the observed term-differences are the residuals

obtained by subtracting the writer's calculated values from the observed term-differences. In the calculation of the Y_{lj} no attempt was made to represent all the data in the table, but only such as are included within the rectangle ruled off in the table. The fit for the data outside the table is not exact, as might be expected, because with rise in the quantum numbers further Y_{lj} which have not been taken into account come into play.

If the observed term-differences are correct, the fit of the writer's analysis outside the rectangle need not necessarily be exact, but the deviations should vary smoothly in passing from one outside member to the next. The fact that some of the deviations show irregularity supports the wisdom of limiting the analysis to the rectangle, since incorrect edge values would have the effect of distorting the analysis. The object of the analysis is to obtain correct values of the Y_{lj} corresponding to low values of l and j, and the procedure adopted seems likely to produce this result.

Within the rectangle the fit of the calculated term-differences is satisfactory except for the fact that the row of term-differences for $\Delta v = 2 - 1$ tends to give negative residuals, while the row for $\Delta v = 3 - 2$ tends to give positive residuals. It is as though the vibrational level $v = 2$ were about 0·03 cm.$^{-1}$ higher than it ought to be. It is, however, more probable that, if the data had allowed higher Y_{lj} to be estimated, this apparent disturbance of the level $v = 2$ could have been made to disappear.

The values of the Y_{lj} used in determining the calculated term-differences are shown in Table II. The pure-vibration and reciprocal-action Y_{lj} were found by the simple method of fitting the experimental

TABLE II.—Y_{lj} FOR THE $1s\sigma 2p\sigma\,{}^1\Sigma$ STATE OF D_2.

	$j=0.$	1.	2.	3.
$l=0$		10·07039	−0·0042240	0·000001423
1	963·330	−0·42518	0·0004319	
2	−11·0444	0·037303	−0·00003862	
3	0·37672	−0·0040234	0·000001108	
4	−0·028024	0·00028308		
5	0·0009131	−0·0000088818		

term-differences in each column within the rectangle by a regression formula of the form:

$$F(v+1, K) - F(v, K) = \Sigma\{(V+1)^l - V^l\}Z_{lk}$$
$$= Z_{1\kappa} + (2V+1)Z_{2\kappa} + (3V^2+3V+1)Z_{3\kappa} + \ldots, \quad (8)$$

where V takes values $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, . . .

When the values of the various $Z_{6\kappa}$ found in this way were compared, it was noticed that three were positive and three negative. It was accordingly assumed that the coefficient Y_{60} was too small to make itself felt,

or at least too small to produce any effect greater than that produced by the small irregularities in the experimental term-differences. Y_{60} was therefore neglected.

On the assumption that the $Z_{6\kappa}$ were negligible, the best values of the $Z_{5\kappa}$ were found to be all positive, all about the same magnitude, and showing a tendency to decrease from Z_{51} to Z_{56}. It was accordingly assumed that Y_{50} and Y_{51} had had appreciable effect, and the $Z_{5\kappa}$ were fitted by a regression formula:

$$Z_{5\kappa} = Y_{50} + K(K+1)Y_{51}. \qquad (9)$$

The values of Y_{50} and Y_{51} found in this way were taken as the final values, and the $Z_{5\kappa}$ were recalculated to fit equation (9) exactly. The corrected $Z_{5\kappa}$ were then substituted in the least-squares equations for the determination of the Z_{lk}, and the best values of the $Z_{4\kappa}$ extracted and fitted by a regression formula:

$$Z_{4\kappa} = Y_{40} + K(K+1)Y_{41} + K^2(K+1)^2Y_{42}. \qquad (10)$$

Y_{42} calculated by this method was found scarcely to affect the result. It was accordingly assumed to be negligible, and Y_{40} and Y_{41} were determined.

The process was repeated, until finally Y_{10}, Y_{11}, and Y_{12} had been determined from the $Z_{1\kappa}$. All the pure-vibration and reciprocal-action coefficients were then determined, and it only remained to eliminate the effect of the latter from the rotational term-differences in order to obtain the pure-rotation coefficients Y_{01}, Y_{02}, and Y_{03}.

The rotational term-differences for the $1s\sigma 2p\sigma\,{}^1\Sigma$ state of D_2 have been given by Dieke (1936), although Dieke includes in his table values which are not derived from actual band lines, but are built up with the help of relations of the type of equation (7). The Y_{lj} of Table II fit the rotational term-differences satisfactorily except some of the edge values. The data for this state show a few slight inconsistencies, when tested by the relations like equation (7), but are probably the best we have for any hydrogen state.

Table III gives the Y_{lj} determined in a similar manner for the $1s\sigma 2s\sigma\,{}^3\Sigma$ state of D_2. These fit all the term-differences deducible from the lines of the $3p\,{}^3\Pi \rightarrow 2s\,{}^3\Sigma$ band system measured by Dieke and Blue

TABLE III.—Y_{lj} FOR THE $1s\sigma 2s\sigma\,{}^3\Sigma$ STATE OF D_2.

	$j=0$.	1.	2.	3.
$l=0$		17·13040	−0·00586405	0·0000050584
1	1885·837	−0·607696	0·00036662	
2	−35·97800	0·0123764	−0·000113788	
3	0·347212	−0·00095279	0·0000190024	
4	−0·0102897	0·0000329219	0·00000105098	

(1935) very closely except for a few edge values. There is another extensive band system ending on $1s\sigma 2s\sigma^3\Sigma$, viz. the $3p^3\Sigma \rightarrow 2s^3\Sigma$ system measured by Dieke (1935). The writer had already commenced an analysis of the state before the publication of the latter band system, and has allowed his analysis to stand, because the inclusion of data from the second band system would make a small difference to the result.

Tables II and III both show the pattern of alternately positive and negative coefficients familiar for the hydrogen states. In the experimentally determined Y_{lj} of these tables we have sufficient material for an application of Dunham's method of determining the constants of the states.

It is to be noted that, as Dunham has shown, each Y_{lj} is composed of a major term and a series of correction terms each smaller than the last in the ratio $u_e^2 : 1$. As u_e^2 is a small quantity (about 0·0004 for D_2), only the first correction term has appreciable effect. Following Dunham we accordingly write Y_{10} as:

$$Y_{10} = \omega_e\{1 + \beta_{10}(u_e/2)^2 + \ldots\}, \qquad (11)$$

and similarly for the other Y_{lj}. Within the limit of accuracy of the correction terms $u_e/2 = Y_{01}/Y_{10}$, so the correction can be very easily made. According to Dunham's method of carrying out the analysis of a state only β_{01}, β_{10}, and β_{02} are usually required, and these correction coefficients can most conveniently be calculated directly from the Y_{lj}. (The expressions for β_{10} and β_{02} given in equation (19) of Dunham's paper (1932) require alteration. In the expression for β_{10} the sign before the term $5a_1^4/2$ should be a plus. In the expression for β_{02} the last term should be $31Y_{20}^2/18Y_{01}^2$. The writer's previous work on H_2 also requires alteration (1935).)

It is true that Dunham's procedure for calculating the configuration constants is based upon approximations which are not entirely satisfactory, unless u_e has a very small value—smaller than those yielded by the hydrogen isotopes. If we could be sure that our experimentally determined Y_{lj} were exact, it would be possible to determine all the molecular constants exactly from Dunham's theoretical expressions for the Y_{l0} and Y_{l1}. This is, unfortunately, something of which we cannot be sure. An inspection of Tables I and II suggests that for the state under consideration Y_{60} ought to have some effect, although our data are not sufficiently good to make it possible to determine this coefficient. If Y_{60} could be determined, its presence would affect the calculated values of the other Y_{l0}. Y_{50} would thus be considerably affected, Y_{40} to a less extent, and so on. The most we can claim is that the Y_{lj} for which l and j take the lowest values do not differ much from their true values.

Following Dunham we begin by adopting Y_{10} as an approximate value of ω_e and Y_{01} as an approximate value of B_e, and by using these to calculate the configuration constants. Since $(u_e/2)^2$ is approximately 0·0001 for the states of D_2, Y_{10} will give ω_e correctly to the first four figures, while the correction term will affect the fifth figure. Similarly Y_{01} will give B_e correctly to four figures. Y_{11}, from which a_1 is calculated, also agrees to four figures with its own major term. It is evident from an inspection of Dunham's expressions that the value of $1+a_1$ can be found correct to four figures. As this value is less than unity for the hydrogen states, a_1 should be determined essentially correct to the fourth decimal place. A similar argument applies to all the a_n determined by Dunham's method, although by the time we come to a_6 we shall have to reckon with the cumulative effect of small errors as well as with the uncertainty attached to the experimental determination of Y_{40}.

All the constants calculated by Dunham's method for the $1s\sigma 2p\sigma\,{}'\Sigma$ and $1s\sigma 2s\sigma\,{}^3\Sigma$ states of D_2 and H_2 are collected in Table IV. The Y_{lj} for the two states of H_2 are given in the writer's previous paper (1935). The constants of these states, however, require alteration and are here recalculated.

TABLE IV.—CONSTANTS OF THE $1s\sigma 2p\sigma\,{}'\Sigma$ AND $1s\sigma 2s\sigma\,{}^3\Sigma$ STATES OF D_2 AND H_2.

	$1s\sigma 2p\sigma\,{}'\Sigma$.		$1s\sigma 2s\sigma\,{}^3\Sigma$.	
	D_2	H_2	D_2	H_2
a_1	$-1{\cdot}6731_{38}$	$-1{\cdot}6867_{29}$	$-1{\cdot}6508_{85}$	$-1{\cdot}6430_{78}$
a_2	$2{\cdot}7680_{92}$	$2{\cdot}8060_{85}$	$2{\cdot}0066_{16}$	$1{\cdot}9564_{96}$
a_3	$-4{\cdot}0780_{61}$	$-4{\cdot}1813_{06}$	$-2{\cdot}1544_{88}$	$-2{\cdot}0896_{82}$
a_4	$4{\cdot}7876_{56}$	$5{\cdot}3836_{28}$	$1{\cdot}9198_{94}$	$2{\cdot}2680_{82}$
a_5	$-3{\cdot}4605$	$-5{\cdot}7249$	$-1{\cdot}1920$	$-2{\cdot}9779$
a_6	$-1{\cdot}6284$	$2{\cdot}7707$	$0{\cdot}4960$	$4{\cdot}1706$
β_{01}	$-3{\cdot}8857$	$-4{\cdot}1208$	$-0{\cdot}4619$	$-0{\cdot}6242$
β_{10}	$-7{\cdot}4010$	$-5{\cdot}0827$	$-2{\cdot}1674$	$0{\cdot}2531$
β_{02}	$13{\cdot}5475$	$4{\cdot}3256$	$6{\cdot}7797$	$-2{\cdot}4709$
B_e (cm.$^{-1}$)	10·07467	20·05112	17·13105	34·22403
ω_e (cm.$^{-1}$)	964·111	1361·018	1886·174	2665·814
u_e	0·0208994	0·0294649	0·0181649	0·0256762
a_0 (cm.$^{-1}$)	23065·5	23095·6	51918·2	51912·1
r_e (A.U.)	1·29	1·29	0·986	0·986
Y_{02} (calc.) (cm.$^{-1}$)	−0·0044070	−0·0174243	−0·0056558	−0·0225537
Y_{02} (obsd.) (cm.$^{-1}$)	−0·0042240	−0·0176610	−0·0058641	−0·0225064
Percentage discrepancy	−4·3	1·4	3·5	−0·2
$\dfrac{a_0 \text{ for } D_2}{a_0 \text{ for } H_2}$	0·99870		1·00012	
$\dfrac{\text{Moment of Inertia for } D_2}{\text{Moment of Inertia for } H_2}$	1·99025		1·99778	

Perhaps the most strange discrepancy between the isotopic states exhibited by Table IV is the unexpectedly large difference in the configuration constants. An examination of the configuration constants of any one of the four states shows that these are not independent. It is possible to construct simple relations between them involving integral coefficients, a fact which suggests that, although the mathematical law of formation of the potential functions is unknown, it might be possible to guess it from an accurate empirical knowledge of the potential functions of a sufficient number of states. It is proposed to deal with this question in a subsequent paper.

The B_e constants give a way of comparing the moments of inertia of any pair of molecules, since we have:

$$\frac{B_e \text{ for } H_2}{B_e \text{ for } D_2}=\frac{\mu r_e^2 \text{ for } D_2}{\mu r_e^2 \text{ for } H_2}. \qquad (12)$$

Unfortunately the rotating-vibrator model does not give an exact way of comparing the equilibrium internuclear distances in a pair of isotopic states, because the reduced masses (μ) involve the masses of the external electrons. The value of the ratio $\frac{\text{mass of deuteron}}{\text{mass of proton}}$ is now known experimentally with accuracy. If we take Aston's (1936) value of the isotopic weights of the neutral hydrogen atoms on the basis $O^{16}=16$, viz.

$$\left.\begin{array}{l}\text{Mass of hydrogen atom } H^1=1{\cdot}00812 \pm 0{\cdot}00004,\\ \text{Mass of hydrogen atom } H^2=2{\cdot}01471 \pm 0{\cdot}00007,\end{array}\right\} \qquad (13)$$

we have:

$$\frac{\text{Mass of deuteron}}{\text{Mass of proton}}=\frac{2{\cdot}01471-0{\cdot}000547}{1{\cdot}00812-0{\cdot}000547}$$
$$=1{\cdot}99902, \qquad (14)$$

0·000547 being the mass of the electron on the same basis.

The ratio $\frac{\text{mass of electron}}{\text{mass of proton}}$ is, according to Birge (1929), $1/1838{\cdot}26 \pm 1$, *i.e.* 0·000544. We easily find that the ratio of the moments of inertia of the isotopic molecules can be expressed as:

$$\frac{\mu r_e^2 \text{ for } D_2}{\mu r_e^2 \text{ for } H_2}=\left(\frac{r_e \text{ for } D_2}{r_e \text{ for } H_2}\right)^2(1{\cdot}99902-0{\cdot}000272f), \qquad (15)$$

where f is a positive function of position of the external electrons expressed in terms of the internuclear distances. As the quantity on the left-hand side of equation (15) has the value 1·99025 for the $1s\sigma 2p\sigma'\Sigma$ state, it is extremely probable that, for this state at least, $\left(\frac{r_e \text{ for } D_2}{r_e \text{ for } H_2}\right)^2$ is less than unity, *i.e.* that the internuclear distance is greater for H_2 than for D_2.

This conclusion is what we should expect from the fact that a_0 for H_2 exceeds a_0 for D_2. In the case of the $1s\sigma 2s\sigma\,^3\Sigma$ state we cannot draw so definite a conclusion. The equilibrium internuclear distances of the isotopic molecules must, however, differ very little.

A curious discrepancy between the isotopic states is that between the a_0 constants. It has been usual in band isotope theory to treat the isotopes as having the same potential function. This assumption cannot be very correct for very light molecules. Unfortunately a_0 is not a constant about which we know very much. If the internuclear distance in the two isotopic molecules is the same, then we know that the force of repulsion between the nuclei, when not vibrating, viz. e^2/r_e^2, is the same. The force of restitution, when the nuclei of either molecule are given a very small displacement, is $-2a_0(r-r_e)/r_e^2$; but this is an entirely different matter, as this force depends on the binding of the molecule by the external electrons as well as the repulsion of the nuclei. The introduction of the constant a_0 is in fact our confession that we are not able to solve the four-body problem presented by the neutral diatomic molecule. When r_e differs for two isotopic molecules, we should expect a_0 to differ too, but the manner of differing need not be simple.

Dunham's method of testing whether his theoretical expressions for the Y_{ij} give a self-consistent fit is to calculate the coefficient Y_{02} from the theoretical expression:

$$Y_{02} = -B_e u_e^2\{1+\beta_{02}(u_e/2)^2\}, \quad . \quad . \quad . \quad . \quad (16)$$

and to compare the value so obtained with that determined experimentally. Y_{02} is the largest coefficient which has not been used in the calculation of the constants and so is the most convenient for this test. The result of the test for each state is shown in Table IV. The test shows that Dunham's theoretical expressions give a more self-consistent fit for H_2 than for D_2. This is scarcely what we should have expected.

The superiority which the molecular spectrum of D_2 should possess over that of H_2 for the application of Dunham's theory rests in the fact that u_e^2 has half the value that it has for H_2. This means an appreciable increase in the accuracy with which the configuration constants can be calculated, provided the Y_{ij} are found correctly.

The greater accuracy with which the configuration constants can be calculated is, however, compensated by the smaller number of Y_{ij} which the structure of the term-differences allows to be calculated in comparison with H_2. This appears to be the reason why Dunham's analysis gives a more self-consistent result for H_2 than for D_2. It may be that the present available band systems of H_2 with their limited vibrational and rotational development provide an optimum field for the application of the model.

If we had higher band members we should have to introduce so many Y_{lj} that the task would become impossible.

It would be desirable to apply Dunham's theory to some hydrogen-like spectrum with a still smaller value of u_e. Given such a spectrum with a sufficient intensity of the higher band members, a very accurate application of the rotating-vibrator model should be possible. Unfortunately the spectrum of He_2, owing to the instability of the molecule, shows poor vibrational structure.

The result of applying the model to D_2 is sufficiently self-consistent to show that the rotating-vibrator gives a good representation of molecular behaviour for the regular Σ states which have been considered. The fact that it has been possible to apply the model to the two states best represented in the visible spectrum of D_2 and to calculate the constants of the potential functions with some precision opens up new fields of investigation.

SUMMARY.

The conditions under which the rotating-vibrator model may be employed in the study of the spectra of diatomic molecules are considered, and reasons are given for the view that in the regular Σ states of molecules of the hydrogen type the model should give a good representation of molecular behaviour.

Measurements of some of the band systems of D_2 in the visible region of the spectrum are now available from the work of G. H. Dieke and his collaborators. Owing to the greater mass of the D_2 molecule its spectrum should provide better material for the application of the model than that of H_2. Like H_2, however, D_2 gives a spectrum in which lines corresponding to large values of the quantum numbers are not represented with measurable intensity. The result is that it is possible to fit the H_2 spectrum by a term form expressed as an expansion in powers of the quantum numbers with a greater number of coefficients of such powers. The model is consequently found to give a more self-consistent fit for H_2 than for D_2.

Difficulties raised by the practical application of the theoretical work of J. L. Dunham to the spectrum of D_2 are discussed, and the constants of the $1s\sigma 2p\sigma\,{}^1\Sigma$ and $1s\sigma 2s\sigma\,{}^3\Sigma$ states are calculated and compared with those for the corresponding states of H_2. It is shown that the potential functions of the isotopic states differ considerably, and that the equilibrium internuclear distance is greater for H_2 than for D_2 at least in the $1s\sigma 2p\sigma\,{}^1\Sigma$ state.

The results of the analysis are sufficiently self-consistent to indicate

that, in the case of these regular Σ states of the hydrogen isotopes, the rotating-vibrator model provides an accurate means of spectral analysis and opens up new avenues for investigation.

This paper includes extracts from a thesis presented to the University of St Andrews in application for the degree of D.Sc. My thanks are due to Professor H. S. Allen for assistance and useful suggestions. I am also indebted to Dr F. L. Arnot of St Andrews for assistance.

REFERENCES TO LITERATURE.

ASTON, F. W., 1936. "Masses of some Light Atoms measured by means of a New Mass-Spectrograph," *Nature*, London, vol. cxxxvii, p. 357.

BIRGE, R. T., 1929. "Probable Values of the General Physical Constants," *Phys. Rev. Supplement*, vol. i, pp. 1–73.

BRILLOUIN, L., 1926. "La méchanique ondulatoire de Schrödinger; une méthode générale de résolution par approximations successives," *C.R. Acad. Sci. Paris*, vol. clxxxiii, p. 24.

DAVIDSON, P. M., 1932. "Eigenfunctions for Calculating Electronic Vibrational Intensities," *Proc. Roy. Soc.*, A, vol. cxxxv, pp. 459–472.

DIEKE, G. H., 1935. "The $3p^3\Sigma \rightarrow 2s^3\Sigma$ Bands of HD and D_2," *Phys. Rev.*, vol xlviii, p. 606.

—— 1936. "The $2s'\Sigma \rightarrow 2p'\Sigma$ Bands of the Hydrogen Molecule," *Phys. Rev.*, vol. i, p. 797.

DIEKE, G. H., and BLUE, R. W., 1935. "Fulcher Bands of HD and D_2," *Phys. Rev.*, vol. xlvii, p. 261.

DIEKE, G. H., and LEWIS, (Miss) M. N., 1937. "Bands of HD and D_2 Ending on the $2p'\Sigma$ State," *Phys. Rev.*, vol. lii, p. 100.

DUNHAM, J. L., 1932. "The Wentzel-Brillouin-Kramers Method of Solving the Wave Equation," *Phys. Rev.*, vol. xli, p. 713; "The Energy Levels of a Rotating Vibrator," *ibid.*, p. 721.

KRAMERS, H. A., 1926. "Wellenmechanik und halbzahlige Quantisierung," *Zeits. Phys.*, vol. xxxix, p. 828.

RICHARDSON, O. W., 1934. *Molecular Hydrogen and its Spectrum*, Yale Univ. Press.

SANDEMAN, I., 1935. "The Mathematical Representation of the Energy Levels of the Secondary Spectrum of Hydrogen.—II," *Proc. Roy. Soc. Edin.*, vol. lv, pp. 49–61.

VAN VLECK, J. H., and SHERMAN, A., 1935. "The Quantum Theory of Valence," *Rev. Mod. Phys.*, vol. vii, pp. 168–228.

WENTZEL, G., 1926. "Eine Verallgemeinerung der Quantenbedingungen für die Zwecke der Wellenmechanik," *Zeits. Phys.*, vol. xxxviii, p. 518.

(*Issued separately December* 12, 1938.)

II.—The Establishment of the Trichromatic Theory of Colour Vision. By Professor W. Peddie, D.Sc., University College, Dundee.

(MS. received September 21, 1938. Read November 7, 1938.)

THE early years of the nineteenth century saw the final establishment, with regard to the ordinary phenomena of light, of the wave theory. And, in those years, the man, Thomas Young (1807), whose work had been mainly effective towards the final acceptance of that theory, put forward a tentative hypothesis regarding the actions and phenomena of colour vision. It stated essentially that these phenomena were explainable most simply on the assumption that all colour sensations that we can experience are due to the existence of three fundamental colour sensations which can be called into play in various proportions by the action of external luminous stimuli falling upon the retina.

Now, after fully a century of controversy, the establishment of Young's theory is *complete*. Yet controversy has not quite ceased. Its continuation marks the continuation of a peculiar condition which is apt to characterise the development of any theory, but which has been specially pronounced in the case of that of Young.

INTRODUCTION.

Every theory possesses at least one *fundamental* postulate. If such a postulate is contradicted by subsequent experimental evidence, the theory must be abandoned. But there are also, in general, subsidiary assumptions which are made tentatively in order to enable the development, and the testing by experiment, of the theory to proceed. Disproof of one of these postulates does not disprove the fundamental postulate. It contributes to the development of the main theory by indicating the need for, or even the direction of, emendation of the subsidiary assumption which has been found to fail.

Yet, perhaps more than in the case of any other leading physical theory, the recognition by the scientific public of the correctness and sufficiency of the fundamental postulate of the trichromatic theory has been retarded by the confusion of its fate with the fate of a merely tentative, non-essential assumption.

Young's Tentative Suggestion.

In the first place, Young's initial illustrative statement that the main phenomena of colour vision would be accounted for if there existed, in the retina, three sets of nerve endings whose stimulation by light gave rise respectively to the three fundamental sensations, was regarded by some as an essential part of his hypothesis. And the failure to establish anatomically the existence of three different sets of nerve fibres was regarded as destroying the theory.

Helmholtz's Tentative Treatment.

About half a century after Young's discussion had been published, Helmholtz (1860) clearly pointed out the invalidity of this conclusion, and gave another postulate, also merely descriptive and non-essential, regarding the process by which the triplicity was encompassed. There might be three light-sensitive substances whose presence in the otherwise non-differentiated nerve terminals give rise to the triple effect. Here again the absence of confirmation of the existence of the supposed differences was regarded as derogatory to the fundamental postulate. Young's and Helmholtz's own statements were overlooked. This would, no doubt, be due largely to the acceptance of unintentionally incomplete accounts of the theory. The truth or falsity of the theory is not affected by any question of the mode of establishment of the activities.

Young's and Helmholtz's Tentative Treatment of Dichromasy.

Young originally pointed out that the then known phenomena of colour-blindness could most simply be accounted for by the postulate that, in such cases, absence of one of the three actions would give rise to dichromasy. And Helmholtz adopted that explanation explicitly as the simplest possible. Thus, if we call, for example, the three fundamental sensations red, green, and blue respectively, red-blindness would mean green-blue vision.

Ultimately a case of one-eyed red blindness was found. By use of the normal eye, the subject knew the normal colours which he was observing with the abnormal eye. The abnormal spectrum, instead of appearing green-blue, was yellow-blue. This contradiction was viewed by some as giving a proof of the falsity of the theory, whereas it only established the falsity of the arbitrary simplest assumption that dichromasy was caused by absence of one of the activities.

Helmholtz's First Widened Treatment.

But Helmholtz (2nd ed. p. 376) replaced that assumption by the next simplest—that of fusion of two activities. If stimulation, which normally effected the red sensation, unavoidably produced also the sensation of a green colour, and conversely, there would no longer exist two independent sensations but only a single combined sensation corresponding to red-green stimulation, which gives yellow as the resultant sensation. This, along with the complementary unaltered sensation of blue, gives the yellow-blue spectrum occurring in the so-called "red-blind" dichromasy. Similar actions hold in other types.

Helmholtz's Final Treatment.

Even with these specified results made evident, it was quite easy to find cases which did not exactly correspond to them and to suppose therefore that the theory was still insufficient. But Helmholtz (2nd ed. p. 458) did not leave the question in this debatable state. He showed that any colour whatsoever, not lying inside Maxwell's triangle, could be taken as the failing colour, thus giving a dichromatic range literally infinitely wider than the finite, though wide, range made evident in actual observation. All the phenomena of abnormal trichromatic colour vision also come easily within the scope of the theory.

Helmholtz's Treatment of Colour Vectors.

Helmholtz also pointed out that, if we select as three measured quantities the intensity, the hue, and the saturation of a given colour, no other quantity can be found whose magnitude is essential to the reproduction of that colour. And he showed that this simple fact constitutes a demonstration of trichromasy. Three independent variations alone are needed. But full intellectual satisfaction is perhaps not attained until one can express the magnitudes of these three quantities in terms of the magnitudes of three suitably chosen fundamental stimuli, such as those contributed by lights of suitable wave-lengths, by means of which a given colour sensation can be established. This final step was not taken during Helmholtz's lifetime. But he showed distinctly the line of investigation along which progress thereto had to go: and he himself made the first tentative advance.

He showed that colour sensation was entirely analogous to a directed quantity (2nd ed. p. 445). It possesses quality as well as magnitude. Therefore, like a directed line in tridimensional space, it is representable

by a space model in which the three co-ordinate quantities are the intensity components of three suitably chosen standard luminous stimuli. They could also be represented by a model having the values of intensity, hue, and saturation measured along three mutually perpendicular axes. Helmholtz selected the former method as the one dealing with the usual experimental observations.

A very fundamental question arises here. In the case of a directed or vector quantity, it is usual to represent the magnitude by the length of the vector. So, in the case of the vector of colour stimulus, the length of the vector is naturally chosen as representing the intensity of the resultant stimulus due to the three standard component stimuli. This is the *simplest* construction which could be used, and is therefore, on scientific principles, the one which should first have its possibilities explored: and it was therefore tentatively adopted by Helmholtz. The reason why the postulate regarding the construction has to be subjected to test is that the geometrical form of the construction imposes a necessary relation between the resultant intensity and the component intensities. If the tridimensional space is regarded as euclidean, the condition is that the square of the resultant intensity is the sum of the squares of the component intensities; and the truth or otherwise of this consequence has to be settled by experiment. Helmholtz himself expressed grave doubts regarding the result of estimating the equality of the intensities of two differently coloured lights, and this was the necessary basis for the estimation of the component and resultant intensities. He said that in making the comparison he felt that he was dealing with *two* independent variations in sensation, not one only, the two being those of brightness and what, for want of a better term, he might denote as "colour-glow."

Though the law of addition of component intensities in the case of lights of the *same* colour was known to be that of arithmetical addition, it therefore, in his view, was doubtful if that law also held in the case of different colours. In all cases of vector composition the resultant intensity being the square root of the sum of the squares of the components, the only variation that is possible in the geometrical representation lies in the selection of the co-ordinate magnitudes; and these may, so far as theory goes, be any functions of the component intensities.

Helmholtz's Test of his Vector Construction.

As just stated, Helmholtz selected the simplest postulate—that of direct proportionality. And he proceeded to test its validity by having search made for fundamental standards whose employment could give

good agreement between observation and the theoretical curve of differential insensitivity in the spectrum (2nd ed. p. 455). And this was found, by Dr Sell, to be possible to, as Helmholtz said, a really wonderful extent. (It is true that the curves of intensity of each of the three fundamental so-found standards throughout the spectrum showed a marked minimum in the spectral range. But this, in view of Helmholtz's feeling regarding the influence of "colour-glow," could not necessarily be taken as destructive of his postulation of the simplest law of composition. Actually, in order that the characteristic form of the observed curve of differential insensitivity may appear, there must be four points of contrary flexure on the curves of the component intensities throughout the spectrum. In other words, there is an actually observed tendency towards encroachment on the single crest of the curve of component intensity from both sides. The conversion of this crest into a hollow is only an extreme case.) .

Abney's Law and the Vector Construction.

In the further development of the subject, Abney's work on the composition of the component intensities showed that the actual law of composition was that of arithmetical summation, and not that of vector summation as accepted by Helmholtz in his choice of the simplest tentative postulate. His postulate was irreconcilable with Abney's law. Again, the discrepancy has been regarded as destructive of the Young-Helmholtz theory, whereas it merely necessitated the abandonment of Helmholtz's initial tentative assumption, and the substitution for it of the other assumption which would be consistent with Abney's result. Obviously, since the law of vector composition must be retained, the co-ordinates used in the vector space must be those of the square roots of the component intensities. For the vector law then gives as the resultant vector magnitude the square root of the sum of the component intensities, so that the vector length corresponds to the amplitude of vibration in the incident radiation; and the intensity is measured by its square. Energy is always measured by the square of a vector of a directed quantity.

Completion by Pauli and Schroedinger.

This step would undoubtedly have been taken by Helmholtz had he lived. The necessity for it was pointed out by W. Pauli, jun., and the consequent evaluation of the expressions for intensity, hue, and saturation were developed by Schroedinger (1920). But the necessity for trichromasy was made clear by Helmholtz when he pointed out, in the

early days, that there was no other mode of observable difference left between two lights when the differences of intensity, hue, and saturation were made to vanish.

Influence of Threshold Values.

The difficulties felt by many workers in the acceptance of the theory were greatly enlarged by the fact of the intrinsic variability in the sensitiveness of the observing eye. The fundamental threshold values were dependent on many conditions. Correspondence of results were therefore not easy of attainment, and discrepancies were sometimes wrongly attributed to failure of the theory. So far as the theory is concerned, whenever the conditions which influence the thresholds are stated, along with their laws of action, the theory at once applies. Only the details of the application are more complicated.

Controversy has been great regarding the minimum number of fundamental stimuli which are necessary for the complete matching of any colour, but it is now universally admitted that three are always sufficient and in general necessary. Modern colour-mixing apparatus is designed on that basis.

Necessary Triplicity of Sensation.

It has been suggested, however, that although three standard stimuli are sufficient, more than three fundamental sensations may be called into play independently. This idea is mainly psychological in origin. For example, yellow is taken by some as a fundamentally different sensation, although some abnormal observers see it as a mixture of red and green. Yet it is a mathematical certainty that, if the sufficiency of three independent standard stimuli be admitted, the sufficiency of three independent fundamental sensations follows of necessity. For the sensations are called into play by the stimuli, and are therefore representable mathematically as functions of the three stimuli. But there can be no fourth independent function of three variables if three such are given. For, by means of four functions the variables can be eliminated, so giving the fourth as a determined function of the other three. That is to say, it cannot be independent. And so the trichromatic theory of colour sensation is now a definitely established fact. It has ceased to be a mere theory.

Various forms of colour-vision theory have been studied involving the use of four or more component stimuli. Since three only are necessary,

one or more relations amongst the four or more must exist, and it is usual to find these written down. With three only we have the simplest expression, and therefore the one which invites universal use.

There is no more beautiful, self-contained, and instructive example of the cautious development of a theory than that which we find here. Unfortunately no full discussion of Helmholtz's great contribution to it is available to English readers, and the originally published account has long been out of print. Many points of interest are unavoidably passed over in the above outline. I venture to hope that the gap may be filled before long.

REFERENCES TO LITERATURE.

HELMHOLTZ, H. VON, 1860. *Handbuch der Physiologischen Optik*, 1st ed. 2nd sect. 1860: 2nd ed. p. 349; 2nd ed. p. 376; 2nd ed. p. 458; 2nd ed. p. 445; 2nd ed. p. 455.

SCHROEDINGER, E., 1920. "Grundlinien einer Theorie d. Farbenmetrik im Tagessehen" (III Mitteil.), *Ann. der Physik*, vol. lxiii, pp. 491–500.

YOUNG, THOMAS, 1807. *Lectures on Natural Philosophy*, London.

(*Issued separately December* 26, 1938.)

III.—Importance of Dialysis in the Study of Colloids. V.—Colloidal Gold. VI.—Colloidal Vanadium Pentoxide.

By **B. N. Desai**, B.A., LL.B., M.Sc., Ph.D., D.Sc., **P. M. Barve**, M.Sc., and **Y. S. Paranjpe**, M.Sc. (From the Physical Chemistry Laboratory, Wilson College, Bombay, 7, India.)

(MS. received July 5, 1938. Read November 7, 1938.)

Introduction.

In previous papers by Desai and collaborators (Desai and Borkar, 1933 *a*; Desai and Desai, 1933 *b*; Mankodi, Barve, and Desai, 1936 *a*; and Joshi, Barve, and Desai, 1936 *b*) it has been shown that the changes which are produced in the charge on the colloidal particles during dialysis of ferric hydroxide, thorium hydroxide, prussian blue, and arsenious sulphide sols are not so simple as is usually supposed, and that data for viscosity and stability towards electrolytes do not necessarily give direct information with regard to the charge on the colloidal particles. In the present paper measurements of cataphoretic speed in the presence and absence of electrolytes, stability towards electrolytes, and conductivity of colloidal solutions of gold and vanadium pentoxide which have been dialysed, diluted, allowed to age, and exposed to sunlight to different extents, have been presented and the interpretations given in the previous papers amplified.

V.—COLLOIDAL GOLD.

Experimental.

Gold sol was prepared by Zsigmondy's nucleus method in batches of 100 c.c., and dialysed against distilled water in parchment paper bags which were previously kept soaked in distilled water for 3 to 4 days. The dialysis was carried out in a dark room in order to avoid effect of light as far as possible, and the outer water was changed twice a day.

The cataphoretic speed was determined as described in the above papers, the upper liquid used being an equiconducting solution of KCl. In studying the effects of electrolytes on the colloid, the electrolytes were also added to the upper liquid so as to get the same ionic environment. The difference between the direct and reverse movements of the boundary was never found to exceed 5 per cent. Conductivities

were also determined in the same manner as before. The stability was determined by finding out the amount of electrolyte just necessary to change instantaneously the colour of the sol from ruby red to a definite blue, the volume of the mixture colloid + electrolyte being kept constant throughout.

For ageing experiments sols dialysed for different periods were stocked in resistance glass flasks (covered with black paper) for different periods before taking measurements. For experiments on the influence of light, equal volumes of sol were kept in resistance glass beakers (of same capacity) covered with glass plates, and exposed directly to sunlight for different periods. Any loss due to evaporation was made good by the addition of conductivity water.

All the measurements were made at a temperature of 30° C.

Results and Discussion.

In all the tables the cataphoretic speed (cat. speed) (mean of direct and reverse movements) is corrected for viscosity and expressed in centimetres per second per volt per centimetre $\times 10^5$. In the experiments on the effect of electrolytes on the colloid, the concentration of the electrolyte is expressed in millimoles per litre of the mixture colloid + water + electrolyte. Dilution is expressed in terms of the ratio

$$\frac{\text{volume of the diluted sol}}{\text{volume of the original sol}};$$

the original sol has thus dilution 1. Flocculation values (F.V.) are expressed in millimoles of the electrolyte per litre of the mixture.

Section A. Changes during Dialysis.

The results of these experiments are given in Tables I (*a*) and I (*b*).

Table I (*a*).

Period of Dialysis in Days.	Cat. Speed.	F.V. with NaCl.
3	65·75	19·67
6	92·50	22·00
10	105·25	23·67
13	116·50	26·33
18	128·50	32·00
21	109·50	24·00
24	75·00	21·00
26	53·75	17·32

Table I (*b*).

Period of Dialysis in Days.	Sp. Conductivity $\times 10^6$.
0	240·0
4	163·8
8	53·9
10	33·6
13	24·8
17	16·6
21	11·4
25	5·9

It will be seen that with the progress of dialysis both the cat. speed and stability first increase, reach a maximum, and then decrease, whilst the

conductivity continuously decreases. The same behaviour was observed with $Fe(OH)_3$, $Th(OH)_4$, and prussian blue.

Dialysis may be taken roughly as the reverse of the process of adding peptising electrolyte to the colloid (Desai and co-workers, *loc. cit.*), as is indicated by the data in Table II obtained by adding small amounts of KOH (OH′ ions being responsible for the charge on the colloid) to two samples of gold sol dialysed for different periods. With increase in the concentration of KOH the cat. speed passes through a maximum.

TABLE II.

Sol Dialysed for 2 Days.		Sol Dialysed for 25 Days.	
Concentration of KOH.	Cat. Speed.	Concentration of KOH.	Cat. Speed.
0·00	60·20	0·00	90·75
6·66	74·40	1·32	122·50
11·10	122·20	1·96	131·50
13·32	97·00	3·96	117·25
15·54	65·40	6·66	105·25
37·75	31·20	16·50	57·00
		19·88	32·00

The increase in cat. speed is evidently due to preferential adsorption (the word 'preferential' indicating that the ion is adsorbed in the inner sheet of the double layer, *i.e.* on the surface of the particle, the other ion of the peptising agent remaining away in the outer layer) or chemical adsorption of the similarly charged ion. But with increase in the concentration of the added electrolyte there is an increasing tendency for the *electrical* adsorption of the oppositely charged ion, *i.e.* for the adsorbed similarly charged ions to become 'covered' or 'bound,' thus decreasing the charge. Thus at small concentrations of the electrolyte the net effect will be a rise in the cat. speed due to preferential adsorption of the similarly charged ion unless the oppositely charged ion is strongly adsorbable or multivalent (see section *c*), while at higher concentrations the cat. speed will continuously decrease due to the preponderating influence of electrical adsorption. If the oppositely charged ion has strong adsorption, under favourable conditions there might be even a reversal of charge, as stated by Mukherjee (1921).

In view of the foregoing the influence of dialysis on the cat. speed may be explained as follows. As the concentration of the peptising electrolyte in the sol decreases during dialysis the electrically adsorbed ions will become free and the charge will increase due to this effect. At the same time there will be desorption of the chemically adsorbed ions, the charge consequently tending to decrease; but this will occur chiefly in the later stages of dialysis, when the concentration of the peptising

electrolyte is low, as the adsorption due to chemical affinity is strong. The stage of dialysis at which the maximum cat. speed occurs probably represents the condition when all the electrically adsorbed ions have become free, the decrease in cat. speed in the later stages of dialysis being due to the continued desorption of the chemically adsorbed ions.

According to the chemical view of the origin of charge on the colloid particles (Pauli and Valkó, 1929; Pauli, 1935) it must be assumed that the *initial increase in charge* during dialysis is due to the formation of new complex ions on the surface of the particles or to increase in the dissociation of the ionogenic complex as a result of the decrease in the electrolyte content of the intermicellary liquid. The final decrease in charge might be considered, according to the chemical view, as due to membrane hydrolysis of the complex, whereby alkali metal ions are replaced by hydrogen ions, giving a less ionised complex (Pauli and Russer, 1932). It is, however, considered that this view is untenable, since in certain other cases the amount of the metallic ion in the sol is not negligible even when the charge begins to decrease.

The decrease in the conductivity during dialysis is mainly due to decrease in the amount of electrolyte in the sol. The contribution of the particles seems to be inappreciable, since the conductivity has decreased even in the early stages of dialysis when the cat. speed has increased.

The fact that both the cat. speed and stability have changed in the same manner during dialysis would support the accepted view that the stability is directly related to the magnitude of the charge. This view has, however, not been supported in other cases (Desai and co-workers, *loc. cit.*).

Section B. Influence of Dilution on Sols Dialysed for Different Periods.

The results of these experiments are given in Tables III (*a*) and III (*b*), from which it will be seen that the cat. speed and conductivity invariably decrease with dilution.

The processes of dilution and dialysis are similar in regard to their effect on the peptising electrolyte. One would therefore expect that the cat. speed should first increase and then decrease on dilution for those samples of the sol which have been dialysed for periods shorter than that corresponding to the maximum in the cat. speed dialysis curve of the colloid. Indeed this behaviour has been observed by Desai and co-workers (*loc. cit.*) in the case of $Fe(OH)_3$, $Th(OH)_4$, and prussian blue. It is difficult to account for the difference between these sols and gold sol. The stabilising ions in the case of the hydroxides and prussian

TABLE III (*a*).

Period of Dialysis in Days.	Dilution. 1·0. Cat. Speed.	1·0. F.V. NaCl.	1·2. Cat. Speed.	1·33. Cat. Speed.	2·0. Cat. Speed.	2·0. F.V. NaCl.
3	65·75	19·67	47·50	42·00	35·50	21·33
6	92·50	22·00	74·50	47·75	42·50	24·30
10	105·25	23·66	82·50	65·00	60·75	26·66
13	116·00	26·33	95·50	86·00	70·00	28·10
18	128·50	32·00	103·75	97·50	85·50	37·35
21	109·50	24·10	95·75	76·50	67·50	26·00
24	75·00	21·00	67·00	55·50	50·00	22·47
26	53·75	17·33	43·25	39·50	35·00	20·34

TABLE III (*b*).

Sp. conductivity of water used for dilution $= 2{\cdot}32 \times 10^{-6}$ mho.

Period of Dialysis in Days.	Sp. Conductivity $\times 10^6$. Dilution. 1.	2.	4.	20.	40.
0	240·00	171·10	95·00	24·17	14·98
4	163·80	108·10	62·44	22·12	12·77
8	53·86	30·91	19·72	7·33	5·11
10	33·62	20·53	13·88	7·04	4·86
13	24·83	14·08	9·42	6·80	4·71
17	16·61	8·86	7·86	5·97	4·52
21	11·42	6·24	5·63	5·18	4·45
25	5·89	5·24	4·99	4·68	4·21

blue are chemically akin to the material composing the particles, whereas gold is stabilised by hydroxyl ion. That this in some way explains the difference appears improbable, since vanadium pentoxide sol also shows no maximum although stabilised by vanadate ion.

For samples of the sol dialysed for period equal to or more than that corresponding to the maximum in the cat. speed dialysis curve of colloid, any dilution should result merely in the desorption of the adsorbed ions, and therefore the cat. speed will continuously decrease on dilution. Such a behaviour on dilution has been found by Desai and co-workers (*loc. cit.*) for the long-period dialysed three sols mentioned above, and is now found for the long-period dialysed gold sol, as will be seen from Table III (*a*). Mukherjee and co-workers (1927) also observed a continuous decrease in the cat. speed on dilution of gold sol dialysed for a long period.

On dilution of the sol the conductivity invariably decreases, due mainly to the decrease in the amount of electrolyte in the intermicellar liquid.

According to a commonly accepted view of the relation between cat. speed charge and stability, the latter should decrease on dilution since

the cat. speed decreases (Table III (*a*)). Actually the sol becomes more stable, probably due to the fact that the increase in distance between the colloid particles on dilution decreases considerably the chances of collision (Mukherjee and Sen, 1919).

Section C. Changes in Cat. Speed (of Sols Dialysed for Different Periods) in the Presence of Electrolytes.

A summary of these results is given in Table IV, the initial cat. speed with the same electrolyte varying because samples of sols dialysed for different periods have been used.

TABLE IV.

Electrolyte.	Initial Cat. Speed.	Initial Increase of Cat Speed.	Cat. Speed at which Coagulation begins.	Concentration of Electrolyte at which Coagulation begins.
KOH	60·20	62·00	31·20	37·75
	90·75	40·75	32·00	19·88
KCl	47·38	43·39	40·48	22·20
	82·57	nil	31·74	3·33
K_2SO_4	47·38	46·92	38·64	33·00
	82·57	nil	33·58	4·11
$MgCl_2$	47·38	nil	36·80	2·46
	82·57	nil	29·21	0·41
$MgSO_4$	47·38	nil	35·42	3·89
	82·57	nil	31·28	0·53

There is an initial rise in cat. speed on adding small increasing amounts of electrolytes having univalent coagulating ions, except in one case with KCl and K_2SO_4. With electrolytes having bivalent coagulating ions, however, an initial rise in cat. speed does not occur. These changes in cat. speed may be explained by means of the considerations advanced in Section A. The reason for the absence of initial rise in cat. speed in one case with KCl and K_2SO_4 is not clear, but it is possible that at still smaller concentrations of the electrolytes than those employed here, there might have been an initial rise in cat. speed.

Further, if we consider the first sample of the sol in the presence of KOH, KCl, and K_2SO_4, it will be seen that the initial rise in cat. speed is greatest with KOH and least with KCl. This is due to differences in the adsorbabilities of OH′, SO_4'', and Cl′ ions.

If the values of electrolyte concentration at which coagulation begins are examined, it will be seen that a smaller amount of electrolyte is necessary for coagulation when the coagulating ions are bivalent than

when they are univalent. This is to be expected in view of the greater adsorbability of bivalent as compared with univalent coagulating ions.

It appears from the values of cat. speed given in column 4 of the table that the cat. speed at which coagulation begins is not the same in all cases, the variations being from 29·21 to 40·48. The conception of a critical potential put forward by Powis (1915) is thus not supported by the present data.

Section D. Changes on Ageing of Sols Dialysed for Different Periods.

In Tables V (*a*) and V (*b*) are given the results on ageing of two samples of the sol dialysed for 2 days and 22 days respectively.

TABLE V (*a*).

Sol Dialysed for 2 Days.

Age in Days.	Cat. Speed.	Sp. Conductivity $\times 10^6$.	F.V. with $MgCl_2$.
0	105·80	363·8	5·35
15	101·20	327·8	4·82
27	96·60	321·8	4·29
42	94·30	302·9	3·93
58	92·23	298·1	3·39

TABLE V (*b*).

Sol Dialysed for 22 Days.

Age in Days.	Cat. Speed.	Sp. Conductivity $\times 10^6$.	F.V. with $MgCl_2$.
0	49·97	193·0	3·57
13	46·00	156·0	2·86
23	43·70	156·0	2·50
33	41·86	147·0	1·96
40	41·40	144·8	1·57

For both the short-period and long-period dialysed sols the cat. speed, conductivity, and stability (as determined by the flocculation values for $MgCl_2$) gradually decrease on ageing. Freundlich (1926) has stated that many sols show immediately after their preparation various comparatively rapid changes in their properties, while after some time a more or less stationary state is reached. From the present results it would appear that ageing continues up to at least two months.

On ageing there is generally a tendency for the particles to aggregate. In the present case a fall in cat. speed is observed (*cf.* Mukherjee, Chaudhury, and Bhabak, 1936). This is probably due partly to adsorption of adsorbed oppositely charged ions, which decreases the charge, and partly to redistribution of the ions in the double layer (as the result of the aggregation) in such a manner that the net effect is a decrease in the surface density of charge.

The decrease in conductivity on ageing could be due to (1) aggregation of the particles, thus decreasing their number per unit volume; (2) adsorption of ions on the surface of the particles as mentioned above; and (3) decrease in the cat. speed.

The decrease in stability is probably at least partly due to the aggregation of the particles.

Section E. Changes on Exposure to Sunlight of Sols Dialysed for Different Periods.

The results of these experiments are given in Table VI.

TABLE VI.

Exposure in Hours.	Sol Dialysed for 2 Days. Cat. Speed.	Sp. Conductivity $\times 10^6$.	F.V. with $MgCl_2$.	Sol Dialysed for 22 Days. Cat. Speed.	Sp. Conductivity $\times 10^6$.	F.V. with $MgCl_2$.
0	104·88	363·1	5·35	49·91	193·0	3·57
10	90·03	355·9	4·82	42·32	184·3	3·22
20	85·17	348·8	3·57	39·36	176·9	2·43
35	70·38	343·3	3·29	36·34	167·0	1·87
60	53·59	293·1	3·00	33·58	134·0	1·43

It is clear that both for short-period and long-period dialysed samples the cat. speed, conductivity, and stability gradually decrease as during ageing and the changes can be explained on the same basis. According to Nordenson (1915), on irradiation of a gold sol, the degree of dispersion of the particles considerably increases. From the stability and conductivity results it would, however, appear that, as indicated above, in the present case the degree of dispersion probably decreases.

SUMMARY.

Measurements of cataphoretic speed in the presence and absence of electrolytes, stability towards electrolytes, and conductivity of colloidal solutions of gold which have been dialysed, diluted, allowed to age, and exposed to sunlight to different extents are presented.

It is found that with the progress of dialysis the cat. speed and stability pass through maxima, whereas the conductivity continuously decreases.

The cat. speed and conductivity decrease on diluting the sol, but the stability increases.

The cat. speed generally first increases and then decreases on adding small increasing amounts of electrolytes having univalent coagulating ions, while it continuously decreases with electrolytes having bivalent coagulation ions. The conception of a critical potential is not supported.

On ageing or on exposure to light the cat. speed, conductivity, and stability gradually decrease.

The variations in the cat. speed have been discussed from the point of view of the adsorption of ions present in the intermicellary liquid. The physical theory of the origin of the charge on the colloidal particles is supported by these results.

VI.—COLLOIDAL VANADIUM PENTOXIDE.

EXPERIMENTAL.

Vanadium pentoxide sol was prepared by the method of Biltz. A known weight of ammonium vanadate was mixed with water in a mortar to form a paste. Thereafter an equivalent amount of HCl was added drop by drop, and thoroughly mixed with the paste. The red precipitate obtained was removed to a filter-paper and washed with water until it showed a tendency to pass into colloidal solution. It was then transferred to a flask and vigorously shaken with a small quantity of water until a clear deep-red sol was obtained. The sol was dialysed in the same manner as before. NH_4Cl and vanadate ions were detected in the dialysate.

The cat. speed and conductivity were determined by the methods previously described. Dialysate made equiconducting with the sol by the addition of NH_4Cl was used as upper liquid.

The stability of the sol was determined by ascertaining the amount of an electrolyte necessary (flocculation values—F.V.) to give instantaneous coagulation, *i.e.* to attain the same turbidity as the standard immediately on mixing the sol with the electrolyte.

Viscosity was determined by means of an Ostwald viscometer.

Experiments on ageing and exposure to sunlight were carried out as in previous work.

All the experiments were carried out at a temperature of 30° C.

RESULTS AND DISCUSSION.

The viscosity results are expressed in terms of the viscosity of water, taken as unity at the temperature of the experiment. Other data are expressed in terms of the units previously employed.

Section A. Changes during Dialysis.

The results of these experiments are given in Tables I (*a*) and I (*b*).

TABLE I (*a*).—DILUTE SOL.

Period of Dialysis in Days.	Concentration of V_2O_5 in g./litre.	Cat. Speed.	F.V. with NaCl.	Viscosity.	Sp. Conductivity $\times 10^6$.
0	2·75	28·92	10·01	1·401	490·6
2	2·67	35·84	9·22	1·386	373·4
4	2·21	39·48	9·01	1·350	365·2
6	2·06	30·20	8·00	1·675	360·3
8	1·81	26·71	6·33	1·810	355·9
12	1·78	23·92	5·80	2·078	296·0
18	1·60	..	4·67	Highly viscous	..
20	..	..	..	Jelly formed	..

TABLE I (*b*).—CONCENTRATED SOL.

Period of Dialysis in Days.	Concentration of V_2O_5 in g./litre.	Cat. Speed.	F.V. with NaCl.	Viscosity.
0	18·40	51·76	5·34	3·062
2	17·49	60·45	4·67	2·979
4	16·00	73·45	4·33	2·897
6	15·47	59·76	3·17	3·073
8	14·67	46·51	2·33	3·873
10	13·17	..	1·83	Highly viscous
12	..	..	..	Jelly formed

It will be seen that with the progress of dialysis the concentration of V_2O_5 continuously decreases. This is due to the fact that V_2O_5 is appreciably soluble in water. With the progress of dialysis V_2O_5 passes out in the dialysate, and some of the colloidal particles dissolve to restore the equilibrium in the intermicellary liquid to a certain extent.

With the progress of dialysis the cat. speed first increases, reaches a maximum, and then decreases, as in the case of $Fe(OH)_3$, $Th(OH)_4$, prussian blue, and gold sols. In the present case the $-$ve charge on the colloidal particles is due to chemically adsorbed vanadate ions. The cat. speed also first increases and then decreases on adding small increasing amounts of NH_4VO_3, as will appear from a summary of the results given in Table IV (*a*). The changes in cat. speed with the progress of dialysis can therefore be explained in the same manner as in the case of colloidal gold.

The specific conductivity decreases continuously with the progress of dialysis as in the case of gold sol, and the changes can be explained in the same manner. It should, however, be pointed out that the conductivity for the V_2O_5 sol does not decrease with the progress of dialysis to the same extent as in the case of gold sol, owing to dissolution of the colloid.

The stability continuously decreases with the progress of dialysis although the cat. speed passes through a maximum. The sol therefore behaves like $Fe(OH)_3$, $Th(OH)_4$, and prussian blue in this respect, and the same considerations apply to the absence of parallelism between charge and stability in the early stages of dialysis (*i.e.* before the maximum value of cat. speed is reached) as were advanced in the case of those sols.

For both the concentrated and the dilute sol the viscosity first decreases, reaches a minimum when the charge is at its maximum, and then increases as the charge decreases, the sol actually setting to a gel if the dialysis is carried out for a long period. The fact that the sol sets to a gel on extreme

dialysis would suggest that the hydration of the particles increases appreciably with the progress of dialysis. These results would therefore appear to support the view of Gore and Dhar (1929), according to which, other things being equal, decrease in the electric charge on colloidal particles enhances their hydration, and hence the viscosity of the sol. According to v. Smoluchowski (1916) a sol with particles having a high electric charge should be more viscous than one of the same substance with particles having a low electric charge; the present results do not appear to support this view. It should, however, be pointed out that the viscosity may depend upon more than one factor (Desai and Desai, 1933 *b*).

Section B. Changes Produced by Dilution of Sols Dialysed for Different Periods.

The results of these experiments are given in Tables II (*a*) and II (*b*) and III.

TABLE II (*a*).—DILUTE SOL.

Sp. conductivity of water used for dilution = 2·05 × 10^{-6} mho.

Dilution.

Period of Dialysis in Days.	1.			2.			4.	
	Cat. Speed.	F.V. with NaCl.	Viscosity.	Cat. Speed.	F.V. with NaCl.	Viscosity.	Cat. Speed.	Viscosity.
0	28·92	10·00	1·401	24·56	12·34	1·293	20·18	1·109
2	35·84	9·22	1·386	29·43	10·34	1·244	24·81	1·075
4	39·48	9·00	1·350	32·63	9·84	1·189	28·60	1·052
6	30·20	8·00	1·675	28·20	9·22	1·226	23·73	1·095
8	26·71	6·33	1·810	25·40	8·00	1·479	21·01	1·112
12	23·92	5·80	2·078	20·90	..	1·739	18·53	1·121

TABLE II (*b*).

Sp. Conductivity × 10^6.

Dilution.

Period of Dialysis in Days.	1.	2.	4.	6.	8.
0	490·6	403·8	358·7	288·1	208·9
2	373·4	254·7	228·1	200·9	188·5
4	365·2	250·9	207·2	188·0	176·8
6	360·3	248·3	200·8	182·0	174·3
8	355·9	236·0	195·4	179·2	171·0
10	329·7	221·8	186·8	174·3	169·3
12	296·0	213·8	184·3	172·6	168·0

TABLE III.—CONCENTRATED SOL.

Period of Dialysis in Days.	Dilution. 1.			2.			4.			8.		
	Cat. Speed.	F.V. with NaCl.	Viscosity.	Cat. Speed.	F.V. with NaCl.	Viscosity.	Cat. Speed.	F.V. with NaCl.	Viscosity.	Cat. Speed.	F.V. with NaCl.	Viscosity.
0	51·76	5·34	3·062	42·15	6·67	2·268	34·68	8·20	1·523	26·93	10·67	1·263
2	60·45	4·67	2·979	54·24	6·00	1·945	40·80	7·20	1·369	30·27	9·00	1·171
4	73·45	4·33	2·897	60·65	5·67	1·739	44·16	6·33	1·287	32·17	8·00	1·123
6	59·76	3·17	3·073	40·97	4·73	1·890	32·40	5·17	1·349	28·23	7·00	1·143
8	46·51	2·33	3·873	35·71	3·33	2·648	29·35	4·20	1·817	26·35	6·10	1·478

It will be seen that for both the concentrated and dilute sols the cat. speed decreases on dilution of short-period as well as long-period dialysed sols, as in the case of gold sol. As previously pointed out, it is difficult to account for the absence of a maximum in these and its presence in other instances.

The changes in the conductivity, stability, and viscosity of the sol are similar to those observed for gold sol, and are due to the same causes. It will be observed that parallelism does not exist between the degree of stability and the magnitude of the charge.

Section C. Changes in Cat. Speed (of Sols Dialysed for Different Periods) in the Presence of Electrolytes.

The results of these experiments are given in Tables IV (*a*) and IV (*b*).

TABLE IV (*a*).

Electrolyte.	Initial Cat. Speed.	Initial Increase in Cat. Speed.	Cat. Speed at which Coagulation begins.	Concentration of Electrolyte at which Coagulation begins.
HCl	27·10	12·70	27·70	0·364
	29·50	9·80	30·10	0·200
KCl	27·10	13·00	28·00	1·540
	29·50	14·70	28·10	1·332
NH_4Cl	28·50	15·00	21·60	1·176
	31·40	13·70	23·70	0·888
NH_4VO_3	28·50	14·30	22·10	2·680
	31·50	13·30	25·00	1·544
H_2SO_4	27·10	9·30	28·50	0·555
	29·50	8·60	28·10	0·250
K_2SO_4	27·10	11·30	29·10	3·109
	29·50	7·80	26·40	1·886

TABLE IV (*b*).

Electrolyte.	Sol Dialysed for 2 Days. Concentration.	Sol Dialysed for 2 Days. Cat. Speed.	Sol Dialysed for 10 Days. Concentration.	Sol Dialysed for 10 Days. Cat. Speed.
$BaCl_2$	0·0000	27·10	0·0000	29·50
	0·0055	23·80	0·0055	25·60
	0·0083	27·30	0·0111	24·90
	0·0247	37·50	0·0165	39·20
	0·0495	28·90	0·0222	29·20
$MgCl_2$	0·0000	27·10	0·0000	29·50
	0·0272	25·30	0·0055	27·60
	0·0333	33·80	0·0111	27·60
	0·0444	37·20	0·0222	35·70
	0·0555	29·50	0·0333	30·10
	0·0666	28·50	0·0444	26·10
$MgSO_4$	0·0000	27·10	0·0000	24·50
	0·0222	19·20	0·0088	25·00
	0·0666	28·20	0·0132	28·30
	0·0888	36·80	0·0222	38·30
	0·1110	28·40	0·0298	27·00
			0·0333	26·30

It will be seen from Table IV (*a*) that on adding small increasing amounts of electrolytes having univalent coagulating ions the cat. speed first increases and then decreases. As suggested in previous papers, the initial rise in cat. speed is due to preferential adsorption of the similarly charged ions, while the subsequent decrease is due to electrical adsorption of the oppositely charged ions.

The concentration of the different electrolytes at which coagulation begins is also different, as in the case of gold sol.

In the case of all the sols previously examined it has been observed that the cat. speed continuously decreases on adding small increasing amounts of electrolytes having bivalent coagulation ions. From Table IV (*b*) it will, however, be seen that there is an intermediate rise in cat. speed, which may be explained as follows.

In this case also the effect of the preferential adsorption of the similarly charged ions is outweighed by that due to the electrical adsorption of the bivalent cations, so that the cat. speed is initially decreased. The vanadic acid in the intermicellary liquid will, however, form insoluble salts when $BaCl_2$, $MgCl_2$, and $MgSO_4$ are added to the sol (Rabinovitch and Kargin, 1935), leaving a quantity of HCl or H_2SO_4 in the liquid. From Table IV (*a*) it will be seen that the cat. speed first increases and then decreases on adding small increasing amounts of HCl and H_2SO_4. It may therefore be expected that after the initial decrease in cat. speed there should be an increase in the same within a certain range as the

amount of HCl and H_2SO_4 in the sol increases. Finally the cat. speed will decrease due to electrical adsorption of the coagulating ions.

From Table IV (*a*) it will also be seen that the cat. speed at which coagulation begins is in some cases even higher than the initial cat. speed. The conception of a critical potential (Powis, 1915) is thus not supported by these results.

Section D. Changes on Ageing of Sols Dialysed for Different Periods.

The results of these experiments are given in Tables V (*a*) and V (*b*).

TABLE V (*a*).

Sol Dialysed for 2 Days.

Age in Days.	Cat. Speed.	F.V. with $MgCl_2$.	Sp. Conductivity $\times 10^6$.
0	38·60	0·575	167·7
7	37·30	0·500	155·4
14	36·00	0·400	142·0
21	35·10	0·300	133·2
28	33·80	0·260	124·5
39	33·00	0·150	101·3

TABLE V (*b*).

Sol Dialysed for 15 Days.

Age in Days.	Cat. Speed.	F.V. with NaCl.	Sp. Conductivity $\times 10^6$.
0	26·00	4·40	89·5
6	22·00	3·13	79·5
12	20·90	2·50	75·6
18	16·40	1·50	70·5
27	12·80	1·13	67·2

It will be seen that, for both the short-period and long-period dialysed sols, on ageing the cat. speed, stability, and conductivity gradually decrease, as in the case of gold sol, and the behaviour of the two sols may be similarly interpreted.

Section E. Changes on Exposure to Sunlight of Sols Dialysed for Different Periods.

The results of these experiments are given below.

TABLE VI.

	Sol Dialysed for 2 Days.			Sol Dialysed for 15 Days.		
Exposure in Hours.	Cat. Speed.	F.V. with $MgCl_2$.	Sp. Conductivity $\times 10^6$.	Cat. Speed.	F.V. with $MgCl_2$.	Sp. Conductivity $\times 10^6$.
0	49·00	0·65	378·3	30·80	0·35	247·3
5	40·70	0·60	274·4	34·70	0·31	197·3
10	38·20	0·57	256·8	29·20	0·27	123·3
15	30·20	0·51	219·5	25·20	0·24	114·2
25	22·30	0·35	122·5	18·00	0·20	63·5

The cat. speed, stability, and conductivity of the short-period as well as long-period dialysed sols continuously decrease on exposure to sunlight

as during ageing. The changes on exposure can be attributed to factors similar to those postulated in the case of gold sol.

SUMMARY.

Measurements of cataphoretic speed in the presence and absence of electrolytes, stability towards electrolytes, conductivity and viscosity of colloidal solutions of vanadium pentoxide which have been dialysed, diluted, allowed to age, and exposed to sunlight to different extents are presented.

It is found that with the progress of dialysis the conductivity and stability continuously decrease, while the cat. speed first increases and then decreases and the viscosity first decreases and then increases, the maximum in cat. speed and minimum in viscosity occurring at the same stage of dialysis.

On diluting samples of sol dialysed for different periods the cat. speed, conductivity, and viscosity continuously decrease, while the stability increases.

On adding small increasing amounts of electrolytes having univalent coagulating ions, the cat. speed first increases and then decreases. With $BaCl_2$, $MgCl_2$, and $MgSO_4$ the cat. speed first decreases, then increases and again decreases in the presence of large amounts of the electrolytes. The sol coagulates in some cases at a value of cat. speed even higher than the initial cat. speed, so that the conception of a critical potential is not supported.

When the sol is allowed to age or exposed to light, the cat. speed, conductivity, and stability continuously decrease.

The results are interpreted by means of assumptions similar to those employed in the case of the other colloidal systems investigated.

REFERENCES TO LITERATURE.

DESAI, B. N., and BORKAR, S. K., 1933 *a*. "Importance of Dialysis in the Study of Colloids. Part I. Colloidal Ferric Hydroxide," *Trans. Faraday Soc.*, vol. xxix, pp. 1269–1285.

DESAI, B. N., and DESAI, A. K., 1933 *b*. "Importance of Dialysis in the Study of Colloids. Part II. Colloidal Thorium Hydroxide," *Trans. Faraday Soc.*, vol. xxx, pp. 265–271.

FREUNDLICH, H., 1926. *Colloid and Capillary Chemistry* (Eng. Trans.), p. 456.

GORE, V., and DHAR, N. R., 1929. "The Change of Viscosity of Sols and of Precipitating Concentration of Electrolytes with their Purity and the Change of the Ratio of Precipitating Concentrations with the Temperature of Coagulation," *Journ. Indian Chem. Soc.*, vol. vi, pp. 641–652.

JOSHI, C. B., BARVE, P. M., and DESAI, B. N., 1936 *b*. "Importance of Dialysis in the Study of Colloids. Part IV. Colloidal Arsenious Sulphide," *Proc. Indian Acad. Sci.*, vol. iv, pp. 590–602.

MANKODI, G. F., BARVE, P. M., and DESAI, B. N., 1936 *a*. "Importance of Dialysis in the Study of Colloids. Part III. Colloidal Prussian Blue," *Proc. Indian Acad. Sci.*, vol. iv, pp. 480–490.

MUKHERJEE, J. N., 1921. "The Origin of the Charge of Colloidal Particles," *Trans. Faraday Soc.*, vol. xvi, pp. 103–115.

MUKHERJEE, J. N., CHAUDHURY, S. G., and RAI CHAUDHURY, S. P., 1927. "Variation of the Charge of Colloidal Particles. Part II. Effect of Dilution and of Non-electrolytes on the Charge and its Variations with Concentrations of Electrolytes," *Journ. Indian Chem. Soc.*, vol. iv, pp. 493–514.

MUKHERJEE, J. N., CHAUDHURY, S. G., and BHABAK, K., 1936. "Variation of Cataphoretic Velocity of Colloidal Particles during Aggregation. Part I," *Journ. Indian Chem. Soc.*, vol. xiii, pp. 372–389.

MUKHERJEE, J. N., and SEN, N. N., 1919. "Coagulation of Metal Sulphide Hydrosols. Part I. Influence of Distance of the Particles of a Sol on its Stability. Anomalous Protective Action of Dissolved Hydrosulphide," *Journ. Chem. Soc.*, vol. cxv, pp. 461–472.

NORDENSON, H., 1915. "Die Bedeutung des Lichtes für die stabilität kolloider Lösungen," *Zeits. physik. Chem.*, vol. xc, pp. 603–627.

PAULI, WO., 1935. "Some Relations between Electro-chemical Behaviour and the Structure of Colloids," *Trans. Faraday Soc.*, vol. xxxi, pp. 11–27.

PAULI, WO., and RUSSER, ED., 1932. "Die Konstitution des Kolloiden Goldes," *Koll. Zeits.*, vol. lviii, pp. 22–44.

PAULI, WO., and VALKÓ, E., 1929. *Elektrochemie der Kolloide.*

POWIS, F., 1915. "Die Beziehung zwischen der Beständigkeit einer Ölmulsion und der Potentialdifferenz an der Öl-Wassergrenzfläche und die Koagulation kolloider Suspensionen," *Zeits. physik. Chem.*, vol. lxxxix, pp. 186–212.

RABINOVITCH, A. J., and KARGIN, V. A., 1935. "Are Lyophobic Colloids Colloidal Electrolytes?" *Trans. Faraday Soc.*, vol. xxxi, pp. 50–66.

SMOLUCHOWSKI, v., 1916. "Theoretical Considerations with Respect to the Viscosity of Colloids," *Koll. Zeits.*, vol. xviii, pp. 190–195.

(*Issued separately December* 26, 1938.)

IV.—Some Eocene Ostracoda from North-West India. By Mary H. Latham, M.A., Assistant in Geology, Glasgow University. *Communicated by* Lieut.-Colonel L. M. Davies, M.A., F.G.S. (With Fifteen Text-figures.)

(MS. received September 1, 1938. Read November 7, 1938.)

Introduction.

The collections of Eocene Ostracoda described in the following paper were made by Lieut.-Colonel L. M. Davies, R.A., and by Mr Pinfold of the Attock Oil Company. The specimens are of particular interest and importance owing to the fact that very few fossil entomostraca had previously been collected in India.

A small collection of ostracoda from the intertrappean freshwater beds near Nagpur in Central India was specifically identified and described by R. T. Jones (1860, pp. 186–7, pl. x, figs. 71–73), while possible entomostracean remains were referred to by Oldham (1893, pp. 86, 267–8, 272–3) and also in the *Memoirs of the Geological Survey of India for* 1872 (p. 46), but it was not until 1925 that some ostracoda, afterwards identified as *Bairdia subdeltoidea* (Münster), were found in the uppermost Ranikot Beds at Thal by Lieut.-Colonel Davies (1927, p. 265).

Approximately two-thirds of the total number of specimens in the present collection belong to *B. subdeltoidea.*

Both the Bairdiidæ and the Cytheridæ are especially abundant in beds of Upper Ranikot age in the Punjab and Attock Districts. It therefore becomes evident that these beds were of a predominantly estuarine facies. Several of the Cytheridæ are identical with or allied to species from the London Clay of the London Basin. During the later Paleocene and the Lower and Middle Eocene times ostracoda became exceedingly rare and dwarfed, probably owing to the extension of marine conditions.

I am indebted to Colonel L. M. Davies for information about the localities and horizons, and my thanks are also due to Professor A. E. Trueman for helpful criticism and advice concerning the text-figures.

The type and figured specimens are in the British Museum (Nat. Hist.), South Kensington, London.

Description of Species.

Family BAIRDIIDÆ.

Genus Bairdia M'Coy.

Bairdia subdeltoidea (Münster).

Fig. 1.

Cythere subdeltoidea Münster, 1830, p. 64, No. 13.
Cytherina subdeltoidea Roemer, 1838, p. 517, pl. vi, fig. 16.
Bairdia subdeltoidea Jones and Sherborn, 1889, p. 16, pl. i, figs. 15, *a, b*.
" " Blake, 1931, p. 162.
" " Howe, 1934, p. 388, figs. 1, *a, b, c*.

Distribution.—Middle Eocene: Waziristan District.
Lower Eocene: Salt Range, Punjab District; Kohat District.
Paleocene: Salt Range, Punjab District; Attock District; Kohat District; Northern Waziristan District; Waziristan District.

This species has also been identified by Lieut.-Colonel Davies (1927, p. 265) in the uppermost Ranikot Beds (Paleocene) at Thal, N.W. India.

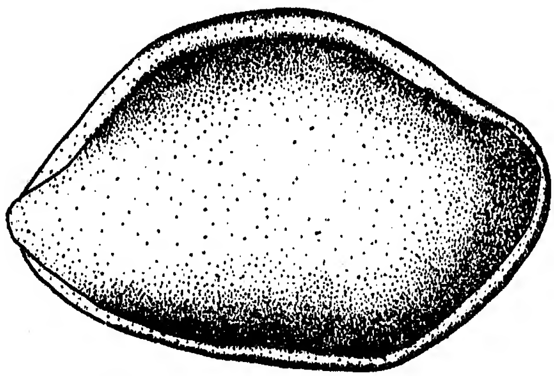


Fig. 1.—*Bairdia subdeltoidea* (Münster). Right valve. (In. 37118.) × approx. 50.

Münster recorded *Bairdia subdeltoidea* from the Tertiary of Osnabruck (type specimen), Paris, Bordeaux, and Castell'arquato, while its occurrence in the Bracklesham *Belosepia* Bed was noted by Jones and Sherborn (1887, p. 452). Howe (1934, p. 388) found numerous specimens of this *Bairdia* in the Red Bluff Beds (Oligocene) at Old Fort St Stephens, Alabama.

Remarks.—Bairdia subdeltoidea is extremely abundant in the present collection from N.W. India. The specimens are well preserved. I have

measured more than two hundred of them and obtained the following results: length, 0·8 mm.–1·9 mm.; height, 0·5 mm.–1·5 mm.; dimensions of average specimens, 1·4 mm. × 0·9 mm. The specimens previously collected by Colonel Davies from Thal averaged 1·6 mm. × 1·0 mm., whereas all except one of Jones and Sherborn's Bracklesham forms were distinctly smaller. Howe gives 0·96 mm. × 0·64 mm. as the average dimensions of *B. subdeltoidea* from the Oligocene of Alabama. In the past many Cretaceous and Tertiary Bairdias were wrongly identified as *B. subdeltoidea*. Blake (1931, p. 162) pointed out that *B. subdeltoidea*, although closely allied to several Cretaceous forms, is really a Tertiary species.

Bairdia cf. *contracta* Jones.

Fig. 2.

Bairdia contracta Jones, 1857, p. 53, pl. v, figs. 1, *a*, *b*, *c*.
,, ,, Davis, 1928, p. 351.

Distribution.—Paleocene: Salt Range, Punjab District; Attock District.

Bairdia contracta was recorded by Jones from the Barton Clay, Hampshire, and by Davis from the London Clay at Clapham.

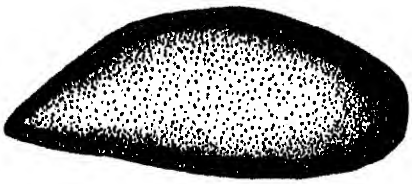


FIG. 2.—*Bairdia* cf. *contracta* Jones. Right valve. (In. 37119.) × approx. 50.

Remarks.— Two specimens of *Bairdia* from the Punjab District and two from the Attock District greatly resemble *B. contracta*. The ventral margin is not so highly arched, however, as it appears to be in Jones's type specimen. The dimensions of the Indian forms are as follows: length, 1·0 mm.–1·1 mm.; height, 0·55 mm.–0·6 mm.

Genus BYTHOCYPRIS BRADY.

Bythocypris subreniformis Jones and Sherborn.

Bythocypris subreniformis Jones and Sherborn, 1887, p. 387.
,, ,, Jones and Sherborn, 1889, p. 16, pl. i, figs. 19, *a*, *b*.

Distribution.—Paleocene: Attock District.

This species was recorded by Jones and Sherborn from the *Belosepia* Bed at Bracklesham.

Remarks.—B. subreniformis is represented by a single specimen (In. 37120) which comes from a bed 100 ft. above the 80-ft. Limestone

on the Kala Chitta Range, Attock District. It agrees closely with Jones and Sherborn's figures.

Family CYTHERIDÆ

Genus CYTHERE Muller.

Cythere cf. *costellata* (Roemer).

Fig. 3.

Cytherina costellata Roemer, 1838, p. 517, pl. vi, fig. 24.
Cythere costellata Jones, 1857, p. 32, pl. v, fig. 11.

Distribution.—Paleocene: Salt Range, Punjab District.

Cythere costellata has been recorded by Roemer from the French Eocene and by Jones from sandy blue clay at Bracklesham.

FIG. 3.—*Cythere* cf. *costellata* Roemer. Left valve. (In. 37121.) × approx. 50.

Remarks.—A single imperfect specimen from Dandot closely resembles Jones's figure of *C. costellata*. The Indian specimen is slightly narrower and is without the denticulations at the anterior and posterior margins, but it has the same number of ribs as the Bracklesham representative.

Cythere ranikotiana n.sp.

Figs. 4A, B, C.

Diagnosis.—Cythere with subovate, inflated carapace. Anterior margin rounded, posterior slightly produced. Dorsal margin straight or slightly convex. Ventral portion of the valves bulging over the ventral edges, which are thus situated in a hollow or depression. Widest and highest at the anterior region. Surface ornamented with delicate reticulations.

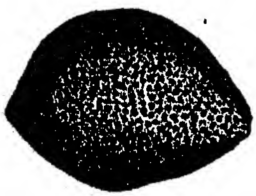


FIG. 4A.—*Cythere ranikotiana* n.sp. Left valve. (In. 37122.) × approx. 40. Holotype.

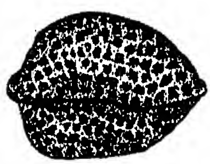


FIG. 4B.—*Cythere ranikotiana* n.sp. Dorsal view. (In. 37123.) × approx. 35.

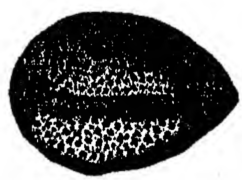


FIG. 4C.—*Cythere ranikotiana* n.sp. Ventral view. (In. 37124.) × approx. 40.

Dimensions.—Length, 0·75 mm.–0·85 mm.; height, 0·5 mm.–0.6 mm.

Distribution.—Paleocene: Salt Range, Punjab District; Attock District; Northern Waziristan District.

Remarks.—Half a dozen specimens of *Cythere ranikotiana* have been found in the fossiliferous shales from Dandot, while two or three additional representatives came from the Attock and Northern Waziristan Districts.

This species of *Cythere* resembles *C. wetherellii* Jones (1854, p. 161, pl. iii, fig. 9) from the Middle Eocene, but the Indian specimens are much more highly inflated and are without the subtriangular impression at the middle of the dorsal portion of each valve.

Genus CYTHEREIS Jones.

Cythereis bowerbanki Jones.

Figs. 5A, B, C.

Cythereis bowerbankiana Jones, 1857, p. 38, pl. vi, figs. 7, 8.
" " Jones and Sherborn, 1887, p. 452, pl. xi, fig. 9.
" " Jones and Sherborn, 1889, p. 34.
" " Davis, 1928, p. 351.
Cythereis bowerbanki Davis, 1936, p. 339.

Distribution.—Middle Eocene: Attock District.
Paleocene: Salt Range, Punjab District; Attock District; Kohat District; Northern Waziristan District; Waziristan District.

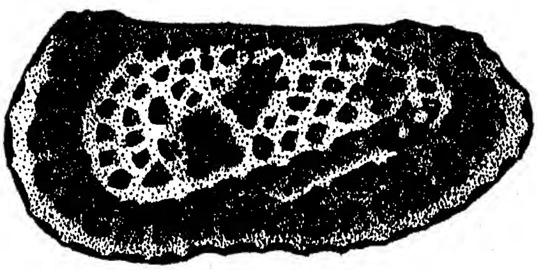


FIG. 5A.—*Cythereis bowerbanki* Jones. Left valve. (In. 37125.) × approx. 60.

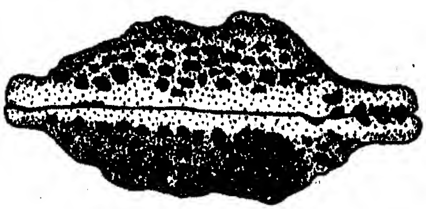


FIG. 5B.—*Cythereis bowerbanki* Jones. Dorsal view. (In. 37126.) × approx. 50.

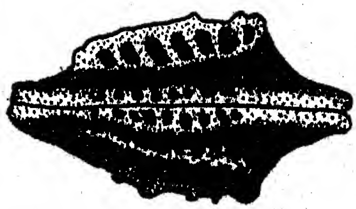


FIG. 5C.—*Cythereis bowerbanki* Jones. Ventral view. (In. 37127.) × approx. 40.

Cythereis bowerbanki was recorded by Jones and Sherborn from the London Clay of Copenhagen Fields, Wimbledon Common, and Whitecliff Bay. It was also identified by Davis in the London Clay at Clapham and at Dorset Road.

Remarks.—About twenty well-preserved specimens of *C. bowerbanki* occur in the Paleocene of N.W. India, and one exceptionally small example comes from the Nummulite Shale Beds of Middle Khirthar age, Attock District.

Cythereis cf. *spiniferrima* Jones and Sherborn.

Fig. 6.

Cythereis spinossissima Jones and Sherborn, 1887, p. 452, fig. 2.
Cythereis spiniferrima Jones and Sherborn, 1889, p. 34, fig. 3.
" " Davis, 1928, p. 351.
" " Alexander, 1934, p. 220, pl. xxxii, fig. 11.
" " Davis, 1936, p. 339.

Distribution.—Paleocene: Salt Range, Punjab District.

Jones and Sherborn found a right valve and a left valve of *C. spiniferrima* in the London Clay, Piccadilly, while Davis identified several specimens of this species in the London Clay at Clapham and at Sheppey. According to Alexander, *C. spiniferrima* is abundant in both the Kincaid and Wells Point formations of the Texan Midway.

FIG. 6.—*Cythereis* cf. *spiniferrima* Jones and Sherborn. Right valve. (In. 37128.) × approx. 60.

Remarks.—A single right valve from the fossiliferous shales at Dandot closely resembles *Cythereis spiniferrima* Jones and Sherborn. In the description of the type specimen, however, it is stated that the anterior area bears, in addition to its marginal row of spines, a second row just within the other. This second row is absent in the Indian specimen, but, as spines of that kind would most certainly be extremely fragile, they may have been accidentally destroyed. Jones and Sherborn do not give dimensions for their specimens, but according to Alexander the Texan representatives are only 0·69 mm. × 0·45 mm. The measurements of the Indian specimens are 1·0 mm. × 0·6 mm.

Cythereis mersondaviesi n.sp.

Figs. 7A, B, C, D.

Diagnosis.—Cythereis with subovate, slightly oblique valves. Dorsal and ventral margins convex. Posterior margin obliquely rounded,

anterior with short beak-like projection. Surface coarsely reticulate, the pits being arranged in lines radiating from near the centre of each valve,

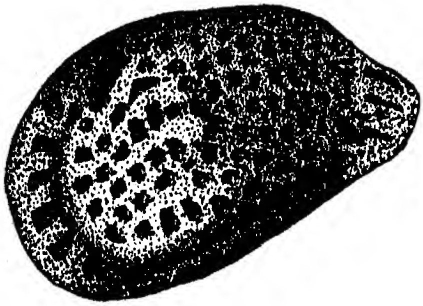


FIG. 7A.—*Cythereis mersondaviesi* n.sp. Right valve. (In. 37129.) × approx. 60. Holotype.

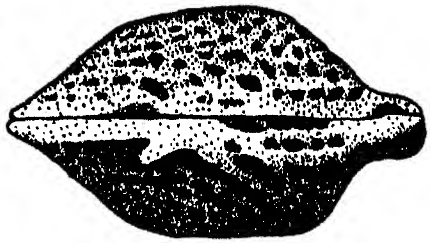


FIG. 7B.—*Cythereis mersondaviesi* n.sp. Dorsal view. (In. 37129.) × approx. 60.

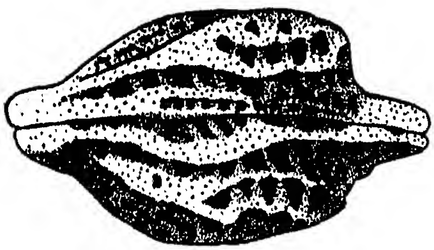


FIG. 7C.—*Cythereis mersondaviesi* n.sp. Ventral view. (In. 37129.) × approx. 60.

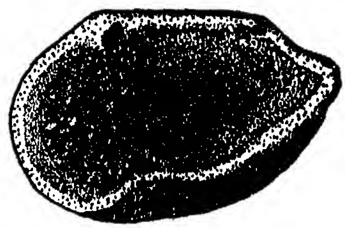


FIG. 7D.—*Cythereis mersondaviesi* n.sp. Interior of left valve. (In. 37130.) × approx. 50.

where an indistinct tubercle is present. Three grooves on anterior projection. Highest at posterior region.

Dimensions.—Length, 0·9 mm.–1·2 mm.; height, 0·6 mm.–0·8 mm.

Distribution.—*Paleocene:* Salt Range, Punjab District; Attock District; Kohat District; Northern Waziristan District; Waziristan District.

Remarks.—Numerous specimens of *Cythereis mersondaviesi* occur in the Paleocene Beds. They are all well-preserved and beautifully ornamented.

Genus CYTHERIDEA Bosquet.

Cytheridea perforata var. *insignis* Jones.

Fig. 8.

Cytheridea perforata var. *insignis* Jones, 1857, p. 46, pl. vi, figs. 3, *a*, *b*, *c*.
" " " " Jones and Sherborn, 1889, p. 39.
" " " " Davis, 1928, p. 351.

Distribution.—Paleocene: Salt Range, Punjab District.

This variety was recorded by Jones from the London Clay, Copenhagen Fields, London, and also by Davis from the London Clay at Clapham.

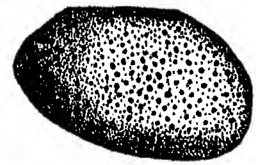


FIG. 8.—*Cytheridea perforata* var. *insignis* Jones. Left valve. (In. 37131.) × approx. 40.

Remarks.—Two well-preserved specimens of *C. perforata* var. *insignis* were found among the ostracoda from the fossiliferous shales at Dandot. They agree perfectly with Jones's figures and the surface pitting is clearly distinguishable.

LIST OF SPECIES AND LOCALITIES.

RANIKOT (PALEOCENE).

SALT RANGE, PUNJAB DISTRICT.

Localities.	*Ostracoda.*
Kala Bagh.	*Bairdia subdeltoidea* (Münster).
Patalla Nullah. High in Patalla Shales.	*Bairdia subdeltoidea* (Münster).
Patalla Nullah. Middle cf Patalla Shales.	*Bairdia subdeltoidea* (Münster).
Nammal Gorge. Bed 6. (Ferruginous limestones overlying the shales.)	*Bairdia subdeltoidea* (Münster).
Fossiliferous shales overlying the Khairabad Limestone at Nammal Gorge.	*Bairdia subdeltoidea* (Münster). *Cythereis mersondaviesi* n.sp.
Fossiliferous shales (almost a pure clay), overlying the Khairabad Limestone at Dandot.	*Bairdia subdeltoidea* (Münster). *Bairdia* cf. *contracta* Jones. *Cythere* cf. *costellata* (Roemer). *Cythere ranikotiana* n.sp. *Cythereis bowerbanki* Jones. *Cythereis* cf. *spiniferrima* Jones and Sherborn. *Cythereis mersondaviesi* n.sp. *Cytheridea perforata* var. *insignis* Jones.
Uppermost layers of Khairabad Limestone at Jaba River.	*Bairdia subdeltoidea* (Münster). *Cythereis bowerbanki* Jones.
Middle of Khairabad Limestone at Chamil.	*Bairdia subdeltoidea* (Münster). *Cythereis bowerbanki* Jones. *Cythereis mersondaviesi* n.sp.
Middle of Khairabad Limestone at Nurpur.	*Bairdia subdeltoidea* (Münster). *Cythereis mersondaviesi* n.sp.
Lower part of Khairabad Limestone at Dhak Pass.	*Bairdia subdeltoidea* (Münster). *Cythereis mersondaviesi* n.sp.
Extreme base of local Eocene at Nammal Gorge.	*Bairdia subdeltoidea* (Münster). *Cythereis mersondaviesi* n.sp.

Attock District.

Bed of Patala Shale age, 375 ft. above 80-ft. Limestone on the Kala Chitta Range.	*Bairdia subdeltoidea* (Münster). *Cythere ranikotiana* n.sp. *Cythereis mersondaviesi* n.sp.
Bed (? Patala Shale age) 100 ft. above 80-ft. Limestone on the Kala Chitta Range.	*Bairdia subdeltoidea* (Münster). *Bairdia* cf. *contracta* Jones. *Cythere ranikotiana* n.sp. *Cythereis bowerbanki* Jones. *Cythereis mersondaviesi* n.sp. *Bythocypris subreniformis* Jones and Sherborn.
Bed (? late Khairabad Limestone age) 30 ft. above 80-ft. Limestone on the Kala Chitta Range.	*Bairdia subdeltoidea* (Münster). *Cythereis bowerbanki* Jones. *Cythereis mersondaviesi* n.sp.
Bed of Khairabad Limestone age just below 80-ft. Limestone on the Kala Chitta Range.	*Bairdia subdeltoidea* (Münster). *Cythereis bowerbanki* Jones. *Cythereis mersondaviesi* n.sp.

Kohat District.

Bed of Khairabad Limestone on the banks of Ishkalai Nullah, Thal.	*Bairdia subdeltoidea* (Münster). *Cythereis bowerbanki* Jones. *Cythereis mersondaviesi* n.sp.
Railway cutting (probably Khairabad Limestone age), Thal.	*Bairdia subdeltoidea* (Münster). *Cythereis mersondaviesi* n.sp.

Northern Waziristan District.

Bed of Khairabad Limestone age, about 12 miles S. of Thal.	*Bairdia subdeltoidea* (Münster). *Cythere ranikotiana* n.sp. *Cythereis bowerbanki* Jones.
Bed about 15 miles S. of Thal.	*Bairdia subdeltoidea* (Münster).
Bed low in upper Ranikot about 9½ miles S. of Thal.	Probably fragment of *Bairdia* sp. indet.

Waziristan District.

Spinkai Scarp.	*Bairdia subdeltoidea* (Münster).
North of Kotkai.	*Bairdia subdeltoidea* (Münster). *Cythereis bowerbanki* Jones. *Cythereis mersondaviesi* n.sp.
North of Kotkai (immediately above bed F950).	*Bairdia subdeltoidea* (Münster).
North of Kotkai (Bed F950, immediately above Bed F944).	*Bairdia subdeltoidea* (Münster). *Cythereis mersondaviesi* n.sp.
North of Kotkai. Bed F944.	*Bairdia subdeltoidea* (Münster).
Bed 1½ miles N. of Kotkai.	*Bairdia subdeltoidea* (Münster). *Cythereis bowerbanki* Jones. *Cythereis mersondaviesi* n.sp.

LAKI (LOWER EOCENE).

SALT RANGE, PUNJAB DISTRICT.

Bhadrar beds overlying the Sakesar Limestone at Sethi.	*Bairdia subdeltoidea* (Münster). *Cythereis bowerbanki* Jones.
Sakesar Limestone at Ratuchia.	*Bairdia subdeltoidea* (Münster).

KOHAT DISTRICT.

Bed of uppermost Laki (Shekhan Limestone) age, at Shadi Khel.	*Bairdia subdeltoidea* (Münster).

KHIRTHAR (MIDDLE EOCENE).

ATTOCK DISTRICT.

Nummulite Shale Beds of Middle Khirthar age, Bhagwan Kas Nullah.	*Cythereis bowerbanki* Jones.

WAZIRISTAN DISTRICT.

Kohat Shale at Spinkai Ghash.	*Bairdia subdeltoidea* (Münster).

REFERENCES TO LITERATURE.

ALEXANDER, C. I., 1934. "Ostracoda of the Midway (Eocene) of Texas," *Journ. Palæont.*, vol. viii, pp. 206–236, pls. xxxii–xxxv, Sharon, Mass.

BLAKE, C. H., 1931. "Notes on Ostracoda. No 4. The Age of *Bairdia subdeltoidea*," *Journ. Palæont.*, vol. v, pp. 162–163, Sharon, Mass.

DAVIES, L. M., 1927. "The Ranikot Beds at Thal (North-West Frontier Provinces of India)," *Quart. Journ. Geol. Soc.*, vol. lxxxiii, pp. 260–290, pls. xviii–xxii, London.

DAVIS, A. G., 1928. "The Geology of the City and South London Railway, Clapham-Morden Extension," *Proc. Geol. Assoc.*, pp. 339–352, London.

——, 1936. "The London Clay of Sheppey and the Location of its Fossils," *Proc. Geol. Assoc.*, pp. 328–345, London.

HOWE, H. V., 1934. "*Bairdia subdeltoidea* (Münster) in the American Tertiary," *Journ. Palæont.*, vol. viii, pp. 388–389, figs. 1, *a*, *b*, *c*, Sharon, Mass.

JONES, T. R., 1854. "Notes on the Entomostraca of the Woolwich and Reading Series," *Quart. Journ. Geol. Soc.*, pp. 160–162, pl. iii, figs. 7–13.

——, 1857. "A Monograph of the Tertiary Entomostraca of England," *Palæontographical Society*, London.

——, 1860. "Note on the Fossil Cypridæ from Nagpur," *Quart. Journ. Geol. Soc.*, vol. xvi, pp. 186–190, pl. x, figs. 71–73.

JONES, T. R., and SHERBORN, C. D., 1887. "Further notes on the Tertiary Entomostraca of England, with Special Reference to those from the London Clay," *Geol. Mag.*, dec. iii, vol. iv, pp. 385–392, 450–460, pl. xi, London.

——, 1889. "A Supplementary Monograph of the Tertiary Entomostraca of England," *Palæontographical Society*, London.

MÜNSTER, G., 1930. "Ueber einige fossile Arten *Cypris* (Muller, Lamk.) und *Cythere* (Muller, Latreille, Desmarest)," *Jahrb. Mineral*, etc., vol. i, pp. 60–67.

OLDHAM, R. D., 1893. *Manual of Geology of India*, 2nd ed., pp. 86, 267–268, 272–273, Calcutta.

ROEMER, F. A., 1838. "Die *Cytherinen* des Molasse Gebirges," *Neues. Jahrb. Mineral*, etc., pp. 514–519, pl. vi.

(*Issued separately December* 26, 1938.)

V.—Solution in Multiple Series of a Type of Generalised Hypergeometric Equation. By Professor T. M. MacRobert, D.Sc.

(MS. received September 27, 1938. Read November 7, 1938.)

§ 1. *Introductory.*—The differential equation

$$\theta(\theta+\rho_1-1)(\theta+\rho_2-1)\ldots(\theta+\rho_p-1)y - x(\theta+\alpha_1)(\theta+\alpha_2)\ldots(\theta+\alpha_{p+1})y=0, \quad (1)$$

where θ denotes the operator xd/dx, has $p+1$ solutions valid for $|x|<1$. These may be denoted by

$$S(p;\, x) \equiv {}_{p+1}F_p(\alpha_r;\, \rho_s;\, x) \quad . \quad . \quad . \quad . \quad (2)$$

and

$$S(p;\, r;\, x) \equiv x^{1-\rho_r}{}_{p+1}F_p\left(\begin{matrix}\alpha_1-\rho_r+1,\ \alpha_2-\rho_r+1,\ \ldots,\ \alpha_{p+1}-\rho_r+1;\ x\\ 2-\rho_r,\ \rho_1-\rho_r+1,\ \ldots\ *\ \ldots,\ \rho_p-\rho_r+1\end{matrix}\right), \quad (3)$$

where $r=1, 2, \ldots, p$, and the asterisk signifies that the expression $\rho_r-\rho_r+1$ is omitted.

The equation has also $p+1$ solutions, each consisting of a power of x multiplied by a hypergeometric function of $1/x$, and valid for $|x|>1$. The connections between these two sets of solutions have been discussed by Thomae (*Math. Ann.*, vol. ii, 1870, pp. 427–444) and also by the present writer (*Phil. Mag.*, ser. 7, vol. xxv, 1938, pp. 848–851).

When $p=1$, solutions involving hypergeometric functions of $1-x$, valid for $|x-1|<1$, also exist. They will be here denoted by

$$\sum(1;\, 1-x) \equiv (1-x)^{\rho_1-\alpha_1-\alpha_2}F\left(\begin{matrix}\rho_1-\alpha_1,\ \rho_1-\alpha_2;\ 1-x\\ \rho_1-\alpha_1-\alpha_2+1\end{matrix}\right), \quad . \quad (4)$$

$$\sum(1;\, 1;\, 1-x) \equiv \frac{1}{\Gamma(\alpha_1+\alpha_2-\rho_1+1)}F\left(\begin{matrix}\alpha_1,\ \alpha_2;\ 1-x\\ \alpha_1+\alpha_2-\rho_1+1\end{matrix}\right), \quad . \quad . \quad (5)$$

and they are connected with the solutions in the domain of $x=0$ by the equations

$$\frac{\Gamma(1-\rho_1)}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)}S(1;\, x)+\frac{\Gamma(\rho_1-1)}{\Gamma(\rho_1-\alpha_1)\Gamma(\rho_1-\alpha_2)}S(1;\, 1;\, x) = \frac{1}{\Gamma(\rho_1-\alpha_1-\alpha_2+1)}\sum(1;\, 1-x), \quad (6)$$

$$\frac{\Gamma(1-\rho_1)}{\Gamma(\alpha_1-\rho_1+1)\Gamma(\alpha_2-\rho_1+1)}S(1;\, x)+\frac{\Gamma(\rho_1-1)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}S(1;\, 1;\, x) = \sum(1;\, 1;\, 1-x). \quad (7)$$

When p is greater than unity, the equations can no longer be solved in terms of hypergeometric series of $1-x$. In the present paper solutions will be found in the form of multiple series involving powers of $1-x$ and valid for $|x-1|<1$. In section 3 a generalisation of (4) is obtained, and in section 4 a number of functions are found which can be regarded as generalisations of (5). Some well-known formulæ required in the discussion are given for reference in section 2.

§ 2. *Some Preliminary Formulæ.*—In addition to (6) and (7), the following formulæ will be assumed:—

$$B(\beta, \rho-\beta)F(\alpha, \beta;\ \rho;\ x)=\int_0^1 t^{\beta-1}(1-t)^{\rho-\beta-1}(1-tx)^{-\alpha}dt, \quad . \quad (8)$$

where $R(\beta)>0$, $R(\rho-\beta)>0$;

$$F(\alpha, \beta;\ \rho;\ x)=(1-x)^{\rho-\alpha-\beta}F(\rho-\alpha, \rho-\beta;\ \rho;\ x); \quad . \quad . \quad (9)$$

and

$${}_{p+2}F_{p+1}(\alpha_r;\ \rho_s;\ x)=\frac{1}{B(\alpha_{p+2}, \rho_{p+1}-\alpha_{p+2})}$$
$$\times\int_0^1 t^{\alpha_{p+2}-1}(1-t)^{\rho_{p+1}-\alpha_{p+2}-1}\,{}_{p+1}F_p(\alpha_r;\ \rho_s;\ xt)dt. \quad (10)$$

From (8) and (7) it follows that, if n is a positive integer and $R(\beta)>0$, $R(\rho-\beta)>0$,

$$\int_0^1 t^{\beta-1}(1-t)^{\rho-\beta-1}(1-tx)^n dt=B(\beta, \rho-\beta)\frac{(\rho-\beta)_n}{(\rho)_n}F\begin{pmatrix}\beta, -n;\ 1-x\\ \beta-\rho+1-n\end{pmatrix}, \quad (11)$$

where $(k)_n\equiv k(k+1)(k+2)\ .\ .\ .\ (k+n-1)$.

§ 3. *A Symmetrical Solution.*—A solution will now be obtained which is symmetrical in the α's and also in the ρ's. It is

$$\sum(p;\ 1-x)\equiv(1-x)^{\sigma_p-s_{p+1}}\prod_{r=1}^{p}\sum_{n_r=0}^{\infty}\frac{(\sigma_r-s_r+\nu_{r-1})_{n_r}(\rho_r-\alpha_{r+1})_{n_r}}{n_r!\,(\sigma_p-s_{p+1}+1+\nu_{r-1})_{n_r}}(1-x)^{n_r}, \quad (12)$$

where $s_r\equiv\alpha_1+\alpha_2+\ .\ .\ .\ +\alpha_r$, $\sigma_r\equiv\rho_1+\rho_2+\ .\ .\ .\ +\rho_r$, $\nu_0=0$, $\nu_r\equiv n_1+n_2+\ .\ .\ .\ +n_r$, $r=0, 1, 2, \ .\ .\ .$

Another useful form of the function is

$$\sum(p;\ 1-x)\equiv\Gamma(\sigma_p-s_{p+1}+1)\prod_{r=1}^{p-1}\sum_{n_r=0}^{\infty}\frac{(\sigma_r-s_r+\nu_{r-1})_{n_r}(\rho_r-\alpha_{r+1})_{n_r}}{n_r!}$$
$$\times\sum_{n_p=0}^{\infty}\frac{(\sigma_p-s_p+\nu_{p-1})_{n_p}(\rho_p-\alpha_{p+1})_{n_p}}{n_p!\,\Gamma(\sigma_p-s_{p+1}+1+\nu_p)}(1-x)^{\sigma_p-s_{p+1}+\nu_p}. \quad (13)$$

The expression for this function in terms of the S-functions can be written

$$K(p)+\sum_{r=1}^{p}K(p;\ r)=\Delta(p), \quad . \quad . \quad . \quad . \quad (14)$$

where

$$\Delta(p) = \frac{1}{\Gamma(\sigma_p - s_{p+1} + 1)} \sum(p;\ 1 - x), \qquad . \quad . \quad . \quad (15)$$

$$K(p) = \frac{\prod_{r=1}^{p} \Gamma(1 - \rho_r)}{\prod_{t=1}^{p+1} \Gamma(1 - a_t)} S(p;\ x) \qquad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and

$$K(p;\ r) = \frac{\Gamma(\rho_r - 1) \prod_{s=1}^{p}{}' \Gamma(\rho_r - \rho_s)}{\prod_{t=1}^{p+1} \Gamma(\rho_r - a_t)} S(p;\ r;\ x), \qquad . \quad . \quad (17)$$

where $r = 1, 2, 3, \ldots, p$.

Formula (14) can be proved by induction. When $p = 1$ it reduces to the known formula (6). Assume that (14) holds for a particular value of p. In it replace x by xt, multiply by

$$t^{a_{p+2}-1}(1-t)^{\rho_{p+1}-a_{p+2}-1},$$

and integrate with regard to t from 0 to 1, it being assumed that $R(a_{p+2}) > 0$, $R(\rho_{p+1} - a_{p+2}) > 0$. Then, on multiplying by

$$\frac{\Gamma(\rho_{p+1})\Gamma(1 - \rho_{p+1})\Gamma(1 - \rho_{p+1} + a_{p+2})}{\Gamma(a_{p+2})\Gamma(1 - a_{p+2})\Gamma(2 - \rho_{p+1})\Gamma(\rho_{p+1} - 1)},$$

we have, by (10),

$$\frac{\Gamma(\rho_{p+1} - a_{p+2})\Gamma(1 - \rho_{p+1} + a_{p+2})}{\Gamma(2 - \rho_{p+1})\Gamma(\rho_{p+1} - 1)} K(p + 1)$$

$$- \sum_{r=1}^{p} \frac{\Gamma(\rho_r - a_{p+2})\Gamma(1 - \rho_r + a_{p+2})\Gamma(\rho_{p+1} - a_{p+2})\Gamma(1 - \rho_{p+1} + a_{p+2})}{\Gamma(\rho_r - \rho_{p+1})\Gamma(1 - \rho_r + \rho_{p+1})\Gamma(a_{p+2})\Gamma(1 - a_{p+2})} K(p + 1;\ r)$$

$$= \frac{\Gamma(1 - \rho_{p+1})\Gamma(1 - \rho_{p+1} + a_{p+2})\Gamma(\rho_{p+1} - a_{p+2})}{\Gamma(1 - a_{p+2})\Gamma(2 - \rho_{p+1})\Gamma(\rho_{p+1} - 1)}$$

$$\times \prod_{r=1}^{p} \sum_{n_r=0}^{\infty} \frac{(\sigma_r - s_r + \nu_{r-1})_{n_r}(\rho_r - a_{r+1})_{n_r}}{n_r!}$$

$$\times \frac{1}{\Gamma(\sigma_p - s_{p+1} + 1 + \nu_p)} F\left(\begin{matrix} s_{p+1} - \sigma_p - \nu_p,\ a_{p+2};\ x \\ \rho_{p+1} \end{matrix}\right). \qquad . \quad . \quad . \quad . \quad . \quad (18)$$

On applying (6) the expression on the right of this equation becomes

$$\frac{\Gamma(\rho_{p+1} - a_{p+2})\Gamma(1 - \rho_{p+1} + a_{p+2})}{\Gamma(2 - \rho_{p+1})\Gamma(\rho_{p+1} - 1)} \cdot \Delta(p + 1);\ x - \frac{1}{\Gamma(1 - a_{p+2})} f(\rho_p, \rho_{p+1}), \quad (19)$$

where

$$f(\rho_p, \rho_{p+1}) = \frac{\Gamma(1-a_{p+2})\Gamma(1-\rho_{p+1}+a_{p+2})}{\Gamma(2-\rho_{p+1})}$$

$$\times \prod_{r=1}^{p} \sum_{n_r=0}^{\infty} \frac{(\sigma_r - s_r + \nu_{r-1})_{n_r}(\rho_r - a_{r+1})_{n_r}}{n_r!}$$

$$\times \frac{1}{\Gamma(\sigma_{p+1}-s_{p+1}+\nu_p)} x^{1-\rho_{p+1}} F\left(\begin{matrix} s_{p+1}-\sigma_{p+1}+1-\nu_p, \; a_{p+2}-\rho_{p+1}+1; \; x \\ 2-\rho_{p+1} \end{matrix}\right)$$

$$= \frac{1}{\Gamma(\sigma_{p+1}-s_{p+1})} \prod_{r=1}^{p-1} \sum_{n_r=0}^{\infty} \frac{(\sigma_r - s_r + \nu_{r-1})_{n_r}(\rho_r - a_{r+1})_{n_r}}{n_r!}$$

$$\times x^{1-\rho_{p+1}} \int_0^1 t^{a_{p+2}-\rho_{p+1}}(1-t)^{-a_{p+2}}(1-xt)^{\sigma_{p+1}-s_{p+1}+\nu_{p-1}-1}$$

$$\times F\left(\begin{matrix} \sigma_p - s_p + \nu_{p-1}, \; \rho_p - a_{p+1}; \; 1-xt \\ \sigma_{p+1}-s_{p+1}+\nu_{p-1} \end{matrix}\right) dt$$

by (8), provided that $R(a_{p+2}-\rho_{p+1}+1) > 0$, $R(1-a_{p+2}) > 0$.

The function $f(\rho_p, \rho_{p+1})$ is symmetrical in ρ_p and ρ_{p+1}. This can be seen by applying formula (9) to the hypergeometric function at the end of the formula just obtained. The function $\Sigma(p+1; \; 1-x)$ is also symmetrical in ρ_p and ρ_{p+1}. This can be proved by writing it in the form

$$\frac{\Gamma(\sigma_{p+1}-s_{p+2}+1)}{\Gamma(1-a_{p+2})}(1-x)^{\sigma_{p+1}-s_{p+2}}$$

$$\times \prod_{r=1}^{p-1} \sum_{n_r=0}^{\infty} \frac{(\sigma_r - s_r + \nu_{r-1})_{n_r}(\rho_r - a_{r+1})_{n_r}}{n_r!}$$

$$\times \frac{1}{\Gamma(\sigma_{p+1}-s_{p+1}+\nu_{p-1})} \int_0^1 t^{\sigma_{p+1}-s_{p+1}-1}(1-t)^{-a_{p+2}}\{1-t(1-x)\}^{a_{p+2}-\rho_{p+1}}$$

$$\times \{t(1-x)\}^{\nu_{p-1}} F\left\{\begin{matrix} \sigma_p - s_p + \nu_{p-1}, \; \rho_p - a_{p+1}; \; t(1-x) \\ \sigma_{p+1}-s_{p+1}+\nu_{p-1} \end{matrix}\right\} dt,$$

where $R(\sigma_{p+1}-s_{p+1}) > 0$, $R(1-a_{p+2}) > 0$, and then applying (9).

Now interchange ρ_p and ρ_{p+1} in (18) and (19) and subtract; thus

$$K(p+1)\left\{\frac{\sin \pi\rho_p}{\sin \pi(\rho_p - a_{p+2})} - \frac{\sin \pi\rho_{p+1}}{\sin \pi(\rho_{p+1}-a_{p+2})}\right\}$$

$$+ \sum_{r=1}^{p-1} K(p+1; \; r)\left\{\frac{\sin \pi(\rho_r-\rho_p)\sin \pi a_{p+2}}{\sin \pi(\rho_r - a_{p+2})\sin \pi(\rho_p - a_{p+2})} - \frac{\sin \pi(\rho_r - \rho_{p+1})\sin \pi a_{p+2}}{\sin \pi(\rho_r - a_{p+2})\sin \pi(\rho_{p+1}-a_{p+2})}\right\}$$

$$- K(p+1; \; p)\frac{\sin \pi(\rho_p - \rho_{p+1})\sin \pi a_{p+2}}{\sin \pi(\rho_p - a_{p+2})\sin \pi(\rho_{p+1}-a_{p+2})}$$

$$+ K(p+1; \; p+1)\frac{\sin \pi(\rho_{p+1}-\rho_p)\sin \pi a_{p+2}}{\sin \pi(\rho_{p+1}-a_{p+2})\sin \pi(\rho_p - a_{p+2})}$$

$$= \Delta(p+1)\left\{\frac{\sin \pi\rho_p}{\sin \pi(\rho_p - a_{p+2})} - \frac{\sin \pi\rho_{p+1}}{\sin \pi(\rho_{p+1}-a_{p+2})}\right\}.$$

But

$$\sin \pi\rho_p \sin \pi(\rho_{p+1} - \alpha_{p+2}) - \sin \pi\rho_{p+1} \sin \pi(\rho_p - \alpha_{p+2}) = \sin \pi(\rho_{p+1} - \rho_p) \sin \pi\alpha_{p+2},$$

and

$$\sin \pi(\rho_r - \rho_p) \sin \pi(\rho_{p+1} - \alpha_{p+2}) - \sin \pi(\rho_r - \rho_{p+1}) \sin \pi(\rho_p - \alpha_{p+2}) = \sin \pi(\rho_r - \alpha_{p+2}) \sin \pi(\rho_{p+1} - \rho_p).$$

Hence (14) holds for all positive integral values of p.

The restrictions on the constants may now be removed.

It follows from (14) that $\Sigma(p;\ 1-x)$ is a solution of the differential equation and that it is symmetrical in the ρ's as well as in the α's.

An alternative form for $\Sigma(p;\ 1-x)$ can be obtained by replacing α_r by $\alpha_r - \rho_p + 1$, $r = 1, 2, 3, \ldots, p+1$; ρ_r by $\rho_r - \rho_p + 1$, $r = 1, 2, \ldots, p-1$; and ρ_p by $2 - \rho_p$: then $S(p;\ x)$ becomes $x^{\rho_p - 1}S(p;\ p;\ x)$; $S(p;\ p;\ x)$ becomes $x^{\rho_p-1}S(p;\ x)$; and, for $r = 1, 2, \ldots, p-1$, $S(p;\ r;\ x)$ becomes $x^{\rho_p - 1}S(p;\ r;\ x)$.

Hence, on multiplying by $x^{1-\rho_p}$, we have

$$\sum(p;\ 1-x) = x^{1-\rho_p}(1-x)^{\sigma_p - s_{p+1}} \times \prod_{r=1}^{p-1} \sum_{n_r=0}^{\infty} \frac{(\sigma_r - s_r + \nu_{r-1})_{n_r}(\rho_r - \alpha_{r+1})_{n_r}}{n_r!\ (\sigma_p - s_{p+1} + 1 + \nu_{r-1})_{n_r}}(1-x)^{n_r} \times \sum_{n_p=0}^{\infty} \frac{(\sigma_{p-1} - s_p + 1 + \nu_{p-1})_{n_p}(1 - \alpha_{p+1})_{n_p}}{n_p!\ (\sigma_p - s_{p+1} + 1 + \nu_{p-1})_{n_p}}(1-x)^{n_p}. \qquad (20)$$

Other forms can then be deduced by interchanging the ρ's.

§ 4. *Non-symmetrical Solutions.*—The complete solution of the differential equation in the domain of $x=1$ requires other p independent integrals. The following p independent solutions are symmetrical in the α's but not in the ρ's. They are denoted by

$$\sum(p;\ r;\ 1-x), \qquad r = 1, 2, \ldots, p,$$

where

$$\sum(p;\ r;\ 1-x) = \left\{ \prod_{s=1}^{p}{}' \Gamma(\rho_s - \rho_r + 1) \Big/ \prod_{s=2}^{p} \Gamma(\alpha_{s+1} - \rho_r + 1) \right\} \times \sum_{n_1=0}^{\infty} \frac{(\alpha_{p+1})_{n_1}}{n_1!}(1-x)^{n_1} \prod_{t=2}^{p-1} \sum_{n_t=0}^{\infty} \frac{(\alpha_{p-t+2})_{\nu_t}(\rho_{p-t+2} - \alpha_{p-t+3})_{n_t}}{n_t!\ (\rho_{p-t+2})_{\nu_t}} \times \sum_{n_p=0}^{\infty} \frac{(\alpha_1)_{\nu_p}(\alpha_2)_{\nu_p}(\rho_2 - \alpha_3)_{n_p}}{n_p!\ \Gamma(\alpha_1 + \alpha_2 - \rho_1 + 1 + \nu_p)(\rho_2)_{\nu_p}}. \qquad (21)$$

They are connected with the solutions in the domain of $x=0$ by the equations

$$\frac{\Gamma(1-\rho_r)\prod_{s=1}^{p}{}'\Gamma(\rho_s - \rho_r + 1)}{\prod_{t=1}^{p+1}\Gamma(\alpha_t - \rho_r + 1)}S(p;\ x) + \frac{\Gamma(\rho_r - 1)\prod_{s=1}^{p}{}'\Gamma(\rho_s)}{\prod_{t=1}^{p+1}\Gamma(\alpha_t)}S(p;\ r;\ x) = \sum(p;\ r;\ 1-x), \qquad (22)$$

where $r = 1, 2, 3, \ldots, p$. From these equations it is clear that the p functions are independent. When $p = 1$, (21) reduces to (5) and (22) to (7).

The general formulæ may be proved by the method employed in the previous section. On integrating in the same way as before and then multiplying by

$$\frac{\Gamma(\rho_{p+1})\Gamma(\rho_{p+1} - \rho_r + 1)}{\Gamma(a_{p+2})\Gamma(\rho_{p+1} - a_{p+2})\Gamma(a_{p+2} - \rho_r + 1)}$$

we obtain the L.H.S. of (22) with $p + 1$ in place of p. Then, making use of formula (11) on the R.H.S., and rearranging the series obtained in powers of $1 - x$, we obtain the formula for $\Sigma(p + 1;\ r;\ x)$. Thus (22) is established.

Note.—Other solutions can be derived by transformations of the constants similar to those employed at the end of section 3.

(*Issued separately April* 12, 1939.)

VI.—**Investigation of Visual Threshold Values.** By **Mrs E. M. Beattie.** *Communicated by* Professor W. PEDDIE, D.Sc., University College, Dundee. (With Five Text-figures.)

(MS. received October 18, 1938. Read December 5, 1938.)

OBJECT OF THE EXPERIMENTS.

THE main object of the experiments was to obtain an indication of the manner in which fatigue of the eye by exposure to illumination of a given intensity, for various periods of time, affected the response of the eye to a rapidly intermittent illumination of constant strength. Expressed otherwise, the object may be stated to be that of determining the effect of fatigue upon the threshold values as made apparent in the flicker test.

APPARATUS.

A flickering source of light was obtained by the use of a rotating disc with equal closed and open sectors in the path of a steady source of light. The disc had eight apertures, which were cut in a strong metal plate. It was spun by an electric motor, the speed of which was recorded on a tachometer and was regulated by a rheostat. The steady beam of light falling on the disc was produced by an electric lamp, L_1 (Opal Ediswan, 200 v., 100 w.), contained in a box which was light-tight apart from a circular aperture (diam. 3·2 cm.) in one of the vertical sides, and the lamp was fixed centrally opposite this aperture. A similar box contained another electric lamp for use as a fatiguing light. A metronome beating seconds was used for the measurement of time intervals.

METHOD OF OBSERVATION.

Preliminary measurements were made on a photometer bench. Using various standard lamps whose candle-power at 200 v. were given by the manufacturers, and a Lummer-Brodhun Head, a mean value for the candle-power of the 100-watt lamp (L_1) was found to be 83·73 candle-power at 200 v. Observations were then made to obtain fig. 1. After the eye had been fully dark-adapted, the disc was set in rotation and the light was observed through an eyepiece and the disc apertures. As quickly as possible the speed of the disc was then increased by adjustment of the rheostat until, at a critical value of the speed, the sensation

of flicker just disappeared. This process was repeated for distances 150, 140, . . ., 100 cm. of L_1 from the eye, which was dark-adapted for *each* reading. The results showed that there was, on any one day, an approximately constant difference between the critical frequencies of intermission (n) of the light at, say, the 150 and the 100 cm. distances. But the values obtained for n in any one set of readings, say, the 100 cm. set, varied from day to day. This seemed to be due to physiological causes. The readings tended to be high on a bright day and low on a dull day.

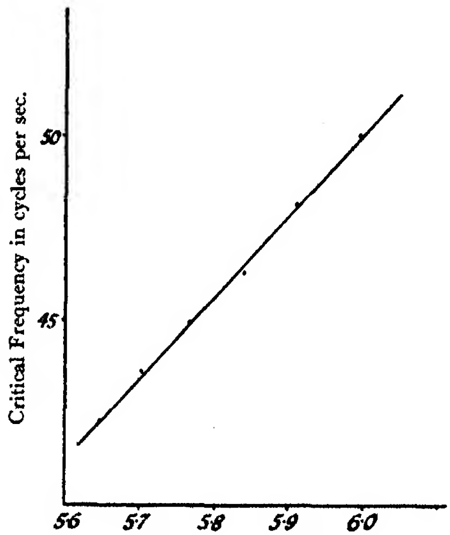


FIG. 1.—$\log_{10} 1/d^2$.

The averages of a series of readings, extending over about twenty days, were taken to get the values of n, the critical frequency, at each distance d between the lamp and the eye. After some practice, the time taken to adjust the speed of rotation of the disc to give the critical frequency could be kept almost constant, so that any error in the readings caused by fatigue of the eye by L_1 would be a constant error.

In fig. 1 the critical frequency is plotted against log $1/d^2$. The observed points are situated on, or very close to, a straight line which does not pass through the origin. This verifies the applicability of the well-known Ferry-Porter law

$$n = k \cdot \log I + k',$$

where I is the intensity of illumination and k, k' are constants.

Since a range of from 100 cm. to 150 cm. was the largest possible with the apparatus as arranged, an attempt was made, by introducing a filter in front of L_1, to cut I down so that the effect on n would be the same as that by increasing d, thus giving a larger range of readings for fig. 1. The proportion in which I was cut down by introducing the filter in front of L_1 was found on the photometer bench, and the equivalent value of d was calculated. It was found, however, that these extra readings lay on a straight line having a different slope from the first part of the graph. This was probably due to a slight colouring in the filter. For if, in the use of the filter, I is changed to $a/100 \cdot I$, reducing the illumination by $(100 - a)\%$, we get

$$n = k(\log . a/100 + \log I)$$
$$= k . \log I + \text{constant}.$$

Thus change of the strength of the illumination *merely* would give no change in the slope of the straight line. The use of the filter was therefore discontinued.

First Group of Curves (fig. 2).

The light L_1 was placed in a position corresponding to about the middle region of fig. 1 ($d=130$ cm.). This group was obtained by

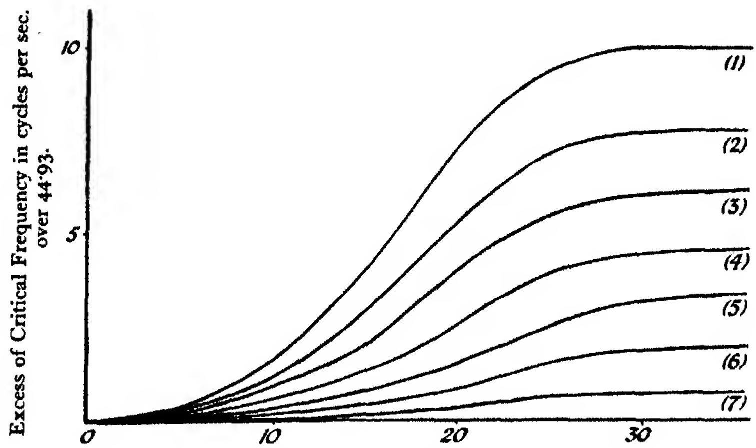


FIG. 2.—Time of fatigue in secs.

fatiguing the dark-adapted eye by a lamp, L_2 (40 w., Opal Ediswan, at 200 v.), for an observed time t (plotted horizontally) and then measuring the critical frequency for L_1 at 130 cm. In the curve (1) of this group, no interval was allowed to elapse between fatiguing the eye and determining n. In the cases of the curves (2), (3), . . ., (7) intervals of 1/2, 1, 3/2, 2, 3, and 5 minutes respectively were allowed to elapse, so that the eye recovered to a corresponding extent from its initial state of fatigue. This interval is subsequently referred to as the "Time of Recovery."

Second Group of Curves (fig. 3).

These were obtained similarly, but in this case the *fatiguing* light was diminished, a 40-watt lamp being used with a ground-glass filter. Curves (1), . . ., (6) correspond to intervals of 0, 1/2, 1, 3/2, 2, and 3 minutes respectively between fatiguing the eye and measuring the critical speed. Table I contains the experimental data used in drawing the curves of fig. 2, and Table II gives similarly the data used for fig. 3.

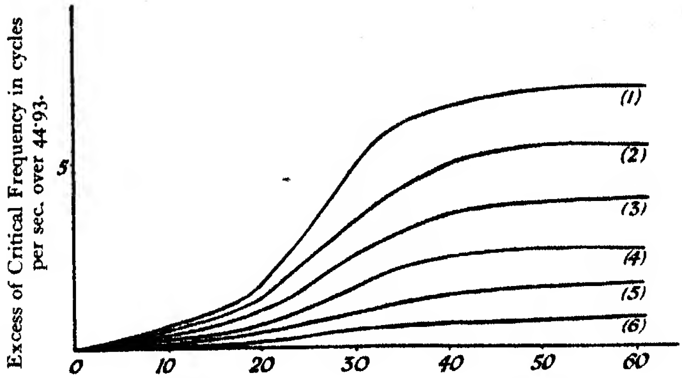


Fig. 3.

Table I.

Time of fatigue in secs. (t).

Time of Recovery in mins. (T).	0.	5.	10.	15.	20.	25.	30.	35.
0·0	0	·4	1·6	4·0	7·3	9·3	10	10
0·5	0	·3	1·2	2·9	5·2	7·1	7·6	7·7
1·0	0	·2	·9	2·0	4·0	5·4	6·0	6·1
1·5	0	·1	·6	1·3	2·5	3·9	4·4	4·5
2·0	0	·1	·4	·8	1·5	2·5	3·2	3·3
3·0	0	..	·26	·4	·8	1·6	1·8	1·9
5·0	0	..	·1	·2	·3	·6	·7	·7

The figures tabulated give ($n-44·93$) cycles per second. That is to say, 44·93 cycles per sec. is the critical frequency for $t=0$, $T=0$.

Table II.

Time of fatigue in secs. (t).

Time of Recovery in mins. (T).	0·0.	10.	20.	25.	30.	40.	50.	60.
0·0	0·0	0·6	1·7	3·1	4·8	6·4	6·8	6·9
0·5	0·0	0·5	1·3	2·3	3·3	4·8	5·3	5·3
1·0	0·0	0·4	1·0	1·6	2·4	3·5	3·8	3·9
1·5	0·0	0·3	0·7	1·1	1·6	2·4	2·6	2·6
2·0	0·0	..	0·5	0·7	0·9	1·4	1·6	1·7
3·0	0·0	..	0·2	0·2	0·5	0·6	0·7	0·8

As in the preceding table the data give the values of ($n-44·93$).

In fig. 4 the critical speed is plotted against the time of recovery, T,

the frequency being measured as before, from the value 44·93. Consider the first curve of the series ($T = 25$ secs.). It is found to have approxi-

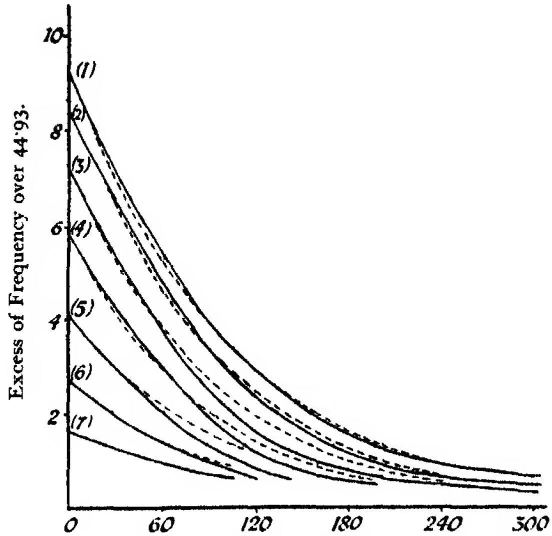


FIG. 4.—Time of recovery in secs.

mately the form of a decaying exponential having

$$n = 44{\cdot}93 + 9{\cdot}3 \exp(-pT).$$

Taking the value of n at

$$T = 120,$$

we have

$$2{\cdot}9 = 9{\cdot}3 \exp(-120p)$$

which gives

$$p = 0{\cdot}009692.$$

Substituting this value in the equation, and plotting the curve so defined, the curve shown by the broken line is obtained. It is seen to lie approximately along the observed curve.

This process was repeated for the remaining curves of the group, and the results are tabulated below.

TABLE III.

Curve.	t.	$(n-44{\cdot}93)$, $T=0$.	$(n-44{\cdot}93)$, T.	T.	p.
1	25	9·3	2·9	120	·009692
2	22·5	8·6	2·4	120	·01064
3	20	7·3	3·8	60	·01087
4	17·5	5·7	2·92	60	·01114
5	15	4·0	2·95	30	·01081
6	12·5	2·65	1·9	30	·01083
7	10	1·0	..	..	..

The values of p are seen to be almost constant, their average value being $p = 0{\cdot}01066$.

The values of the ordinates and abscissæ used in plotting the broken line curves in fig. 4 from the values of $(n - 44{\cdot}93)$ and p, as given in Table III, are contained in Table IV.

TABLE IV.

Values of t in secs.

Curve.	0.	30.	60.	90.	120.	180.	240.
1	9·3	6·95	5·20	3·90	2·90	1·63	0·91
2	8·6	6·25	4·54	3·30	2·40	1·27	0·67
3	7·3	5·27	3·80	2·75	1·98	1·03	0·54
4	5·7	4·08	2·92	2·09	1·50	0·77	..
5	4·0	2·95	2·17	1·60	1·18	..	..
6	2·65	1·90	1·38	1·00	0·72	..	..

The curves of fig. 5 have the same connection with the curves of fig. 3 as those of fig. 4 have with those of fig. 2. The corresponding tables are:

TABLE V.

Curve.	t.	$(n - 44{\cdot}93)$, T = 0.	$(n - 44{\cdot}93)$, T.	T.	p.
1	50	6·8	2·6	90	·01068
2	30	4·8	2·4	60	·01155
3	25	3·1	1·6	60	·01103
4	20	1·7	1·3	30	·00895

TABLE VI.

T in secs.

Curve.		0.	30.	60.	90.	120.	180.
1	$6{\cdot}8 \exp(-{\cdot}01068t)$	6·8	4·94	3·58	2·60	1·89	0·99
2	$4{\cdot}8 \exp(-{\cdot}01155t)$	4·8	3·39	2·40	1·70	1·20	0·60
3	$3{\cdot}1 \exp(-{\cdot}01103t)$	3·1	2·22	1·60	1·15	0·92	..
4	$1{\cdot}7 \exp(-{\cdot}00895t)$	1·7	1·30	0·99	0·76	0·58	..

of the first two figs.

Very laborious calculations were carried out in order to test if formulæ compounded of two, or three, rising or falling exponentials could give a more or less accurate expression for the observed results as shown in figs. 2 and 3, but I found that it was impossible to obtain any such result with simple exponentials. Theoretical reasons might have been given for the applicability of an expression of that type. But, as I was unable to devote more time to the work of calculation, Professor Peddie has had my conclusion confirmed. The inapplicability of these formulæ is due to the very rapid approach of the experimental curves to their asymptotes. It is found that a very close approximation to the observed results is given by the formula

$$y = a(1 - e^{-pt^2}),$$

where a and p are constants. Want of correspondence is apparent mostly at small values of y and t, where the calculated values of y are slightly too small. It was also found that the slightly more complicated formula

$$y = a + (a - bt)e^{-pt^3}$$

gives very accurate representation through the whole of the observed range.

In the case of the curves of fig. 2 the values

$$a = (10{\cdot}2)e^{-(.482)T}$$

are satisfactory; while, in the case of the curves of fig. 3, the values

$$a = (7{\cdot}2)e^{-(.3)T}$$

suit well, T being the time of recovery.

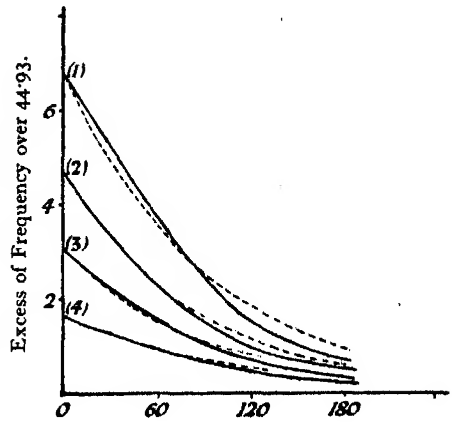


FIG. 5.—Time of recovery in secs.

All these formulæ are, of course, to be regarded as empirical. The closeness of correspondence between observed results and those obtained from the formulæ are shown, for the first curve of fig. 2, in the Table below. At the times t, the observed values are given in the row y_0, the values calculated by the simpler formula above appear in the row y_1, and the values calculated by the more complicated formula are given in the row y_2.

TABLE VII.

t.	5.	6.	10.	15.	20.	25.	30.
y_1	·19	·33	1·43	4·07	7·16	9·24	10·03
y_0	·40	·45	1·60	4·10	7·30	9·30	10·00
y_2	·30	·45	1·61	4·27	7·28	9·29	10·04

The outstanding feature of the curves of figs. 2 and 3 is the increase of response, to subsequent intermittent stimulation, caused by increase of fatigue.

(*Issued separately April* 12, 1939.)

VII.—The Fertility of Scottish Married Women, with Special Reference to the Period 1926–1935.

By **R. S. Barclay**, B.Sc., and **W. O. Kermack**, D.Sc., LL.D. (From the Laboratory of the Royal College of Physicians, Edinburgh.) (With Two Figures and Thirteen Tables.)

(MS. received December 5, 1938. Read March 6, 1939.)

I. Introduction.

In a previous communication (Barclay and Kermack, 1938) it has been shown that the specific legitimate fertility rates of Sweden and Denmark, viewed broadly over several decades, exhibit well-marked regularities, briefly described as conformity to a "diagonal law." The procedure is to express the specific fertility rate, observed for any particular age-group at a certain period, as a percentage of the rate for the same age-group at a time preceding the decline of the birth-rate ("standard rate"). If now these percentages are plotted as contours on a graph, in which the abscissæ represent calendar years and the ordinates women's age, it is found that, when the age-group 15–20 is excluded, the constant percentage curves are approximately straight lines, running parallel to the diagonals in such a sense that, along any line, increase in calendar years corresponds to decrease in women's age. In Finland, the same general effect is apparent, though it is somewhat obscured by minor disturbances. It is found that, in the case of England and Wales, predictions made on the basis of the law give a reasonable agreement with estimated fertility rates calculated by the Registrar-General on the basis of census data.

In the first part of the present investigation, we assume that the diagonal law holds for the specific legitimate fertility rates of Scotland, and, following a method worked out for England and Wales in our previous paper, find a series of specific rates which fit best with the observed facts, namely, the specific legitimate fertility rates for 1855 as calculated by Lewis and Lewis (1906), and the annual numbers of legitimate births for the various census years up to the present. The second part consists of the comparison of the values so found with approximate specific rates obtained by the Registrar-General as the result of a special count made on the census schedules for 1931. The third part presents a general survey of the legitimate fertility of Scotland, 1926–1935,

in the light of the available information. In the final section we discuss some of the points which have arisen during the course of the work.

II. The Application of the Diagonal Law to Norway and Scotland.

Specific legitimate fertility rates for Norway are available for a period as early as 1875–1876, but a regular sequence of quinquennial rates for a period earlier than 1890 does not seem to be available (cf. *Folkemengdens Bevegelse, 1921–1932*). However, this situation affords an excellent opportunity of testing the method developed in our previous paper (pp. 65–69). The rates for the early period may be used as the "standard rates." In addition we have, for each census year, the total number of legitimate births and also the numbers of married women in the various age-groups. We assume that the fall in the specific fertility rates is linear once it has begun (*loc. cit.*, p. 68), and that the diagonal law holds. This means that if F_θ is the standard fertility rate at age θ, then the fertility rate at a time t for this age-group is $F_\theta Q_{t,\theta}$, where $Q_{t,\theta}=p-d(t-t_0+\theta)$, provided this expression is less than unity, otherwise $Q_{t,\theta}=1$. (As explained in the previous paper, the youngest age-group, 15–20, does not come within the diagonal law, but, as the number of births in this category is in any case small, it is sufficient to put $Q_{t,\theta}=1$ for this group, for all values of t.) It is easily seen that this system of theoretical fertility rates is that required by the diagonal law, provided that the diagonals are equally spaced, as is the case for Sweden and Denmark. Two arbitrary constants, p and d, are involved in the above expression, and these must be so determined that the calculated numbers of births agree, as well as possible, with the numbers actually observed.

The theoretical number of these births in the year t is $\sum_\theta M_{t,\theta}F_\theta Q_{t,\theta}$, where $M_{t,\theta}$ is the number of married women aged θ. In practice, it is convenient to work with quinquennial age-groups, and to employ the calendar years 1890, 1900, 1910, and 1930, for which census data are available. It is best not to use the year 1920 in the evaluation of the constants, because in Norway, as in so many countries, the birth-rate in this post-War year was abnormally high.

It is first necessary to decide provisionally, in the case of the earliest census year in question, that is, 1890, how many of the older quinquennial age-groups have already begun to decline in fertility. This is equivalent to deciding for which values of t and θ we take $Q_{t,\theta}=1$, and for which we take $Q_{t,\theta}=p-d(t-t_0+\theta)$.

If now N_t is the actual number of legitimate births in the year t, then, in order to get the maximum agreement, we must determine p and d such that $\sum_t \left[N_t - \sum_\theta M_{t,\theta} F_\theta Q_{t,\theta} \right]^2$ is a minimum. By differentiating with respect to p and d and equating to zero, we obtain the two equations

$$\sum_t \left[N_t - \sum_\theta M_{t,\theta} F_\theta Q_{t,\theta} \right] \left[\sum_\theta M_{t,\theta} F_\theta \right] = 0$$

and

$$\sum_t \left[N_t - \sum_\theta M_{t,\theta} F_\theta Q_{t,\theta} \right] \left[\sum_\theta M_{t,\theta} F_\theta (t - t_0 + \theta) \right] = 0,$$

the summations in both equations being taken over values of t and θ for which $Q_{t,\theta}$ is not equal to unity. These two equations enable us to obtain values for p and d. We then calculate the values of $Q_{t,\theta}$, and verify that the fall did actually begin in the particular quinquennial age-group assumed.

The specific legitimate fertility rates so calculated for Norway are shown in Table I, which also contains the observed rates for the same calendar years. If we exclude the abnormal post-War year 1920, the

TABLE I.—COMPARISON OF OBSERVED AND CALCULATED SPECIFIC LEGITIMATE FERTILITY RATES, NORWAY.

	1890.			1900.			1910.			1920.			1930.		
Age-group.	Observed.	Calculated.	Percentage Error.	Observed.	Calculated.	Percentage Error.	Observed.	Calculated.	Percentage Error.	Observed.	Calculated.	Percentage Error.	Observed.	Calculated.	Percentage Error.
15–	504	*382*		593	*382*		564	*382*		784	*382*		694	*382*	
20–	460	*480*	+4	485	*480*	−1	462	*472*	+2	455	*396*	−13	374	*319*	−15
25–	393	*407*	+4	382	*407*	+7	358	*368*	+3	335	*303*	−10	237	*238*	0
30–	335	*350*	+4	333	*344*	+3	284	*288*	+1	254	*232*	−9	163	*177*	+9
35–	277	*289*	+4	264	*261*	−1	227	*215*	−5	198	*169*	−15	115	*123*	+7
40–	166	*173*	+4	159	*145*	−9	131	*117*	−11	107	*89*	−17	61	*61*	0
45–	36	*36*	0	33	*30*	−9	24	*23*	−4	19	*17*	−11	10	*11*	+10

errors vary from −15% to +10%, with an average error, irrespective of sign, of 4·9%, and a root mean square error of 6·2%. In 16 out of 24 cases the calculated values differ from the observed by not more than 5%. The total numbers of legitimate births to be expected in the various census years, as calculated from the formula $N_t = \sum_\theta M_{t,\theta} F_\theta Q_{t,\theta}$, are shown in Table II, which contains also the observed numbers. The greatest error, −12·27%, occurs in the year 1920, when, as mentioned

above, the birth-rate in Norway was quite abnormal. If this year is excluded, the average error is 1·47 %.

TABLE II.—COMPARISON OF OBSERVED AND CALCULATED LEGITIMATE BIRTHS, NORWAY.

Census Year.	Observed.	Calculated.	Percentage Error.
1890	56,060	58,135	+ 3·70
1900	61,696	61,910	+ 0·35
1910	57,367	56,706	− 1·15
1920	61,793	54,211	− 12·27
1930	43,614	43,906	+ 0·67

These comparisons show that this method of recovering the past specific fertility rates gives very satisfactory results in the case of Norway. It may be emphasised that, in the calculations, only the standard rates (1875–1876) and the total legitimate births for 1890, 1900, 1910, and 1930 were employed, and that the specific fertility rates were therefore calculated quite independently of the observed values, with which comparison is made.

The next step was to apply the same method to the Scottish figures. The numbers of legitimate births occurring during the various census years and the married women distributed in quinquennial age-groups for these years are given in Table III.

TABLE III.—MARRIED WOMEN IN QUINQUENNIAL AGE-GROUPS, AND LEGITIMATE BIRTHS, SCOTLAND, CENSUS YEARS 1861–1931.

Age-group.	1861.	1871.	1881.	1891.	1901.	1911.	1921.	1931.
15–	3,318	3,637	3,434	2,568	3,714	3,273	5,015	4,998
20–	38,942	39,682	46,313	44,219	52,084	47,067	54,657	49,164
25–	68,372	74,574	83,705	85,537	101,694	99,860	102,085	100,790
30–	71,945	78,508	84,177	92,881	107,894	119,953	116,412	123,028
35–	64,239	70,769	79,589	86,918	100,981	117,071	119,260	120,101
40–	58,708	63,827	72,224	75,213	88,221	100,814	114,195	110,197
45–	46,873	51,293	57,604	65,428	72,723	84,870	102,307	102,842
Legitimate Births	97,080	105,051	115,687	116,339	123,833	112,650	114,444	85,559

As standard rates there are available the specific legitimate fertility rates calculated by Lewis and Lewis in their famous investigation of the birth returns for the year 1855, the first year of compulsory birth registration in Scotland, and the only year prior to the passing of the Population (Statistics) Act of 1938 in which the age of the mother was recorded at birth registration. These rates are as follows:—

TABLE IV.—SPECIFIC LEGITIMATE FERTILITY RATES, SCOTLAND, 1855.

Age-group.	Rate (Confinements per 1000 married women).
15–	511·2
20–	427·0
25–	366·0
30–	302·4
35–	242·0
40–	113·3
45–	18·1

A difficulty, however, arises, for, when the above rates are applied to the married women in 1861 and 1871, the numbers of births so calculated are substantially less than the numbers which actually were observed, the deficiency being 9·2% in 1861 and 9·4% in 1871. The recorded crude birth-rates for the years 1855 *et seq.* are given below. There is no

TABLE V.—CRUDE BIRTH-RATES, SCOTLAND, 1855–1864.

Year.	Birth-rate.	Year.	Birth-rate.
1855	31·3	1860	35·6
1856	34·0	1861	34·9
1857	34·3	1862	34·6
1858	34·4	1863	35·0
1859	35·0	1864	35·6

reason to believe that any real rise in the birth-rate occurred during these years, and there seems to be little doubt that the explanation lies in the defective registration, which one might expect to experience during the few years following on the introduction, for the first time, of compulsory birth registration. We find that a similar conclusion has been reached by Newsholme and Stevenson (1906). It is, therefore, necessary to adjust the standard rates, so as to correct for this error. As there is no evidence that the degree of error is correlated with the age of the mother, we have multiplied the rates of Lewis and Lewis by a factor, so chosen that the numbers of births calculated for 1861 and 1871 agreed as well as possible with the numbers actually observed. The 1855 rates were, therefore, multiplied by 1·1, and, in this way, the standard rates found in Table VI were obtained.

TABLE VI.—STANDARD SPECIFIC LEGITIMATE FERTILITY RATES, SCOTLAND.

Age-group.	Fertility Rate.	Age-group.	Fertility Rate.
15–	562	35–	266
20–	470	40–	125
25–	403	45–	20
30–	333		

It was now possible to apply the method used in the case of Norway, and, in this way, it was found that $Q_{t,\theta}=95\cdot6-1\cdot054(t-1923\cdot5+\theta)$, where t is the calendar year and θ is the mean value of the age-group. The specific legitimate fertility rates calculated on the basis of the formula $F_{t,\theta}=F_{\theta}Q_{t,\theta}$ are shown in Table VII.

TABLE VII.—CALCULATED SPECIFIC LEGITIMATE FERTILITY RATES, SCOTLAND.

Age-group.	1881.	1891.	1901.	1911.	1921.	1931.
15–	562	562	562	562	562	562
20–	470	470	449	400	350	301
25–	403	403	364	322	279	237
30–	333	318	283	248	213	178
35–	266	240	212	184	156	128
40–	120	106	93	80	67	54
45–	18	16	14	12	10	8

Calculation of the expected numbers of legitimate births for the various census years gave the results reproduced in Table VIII.

TABLE VIII.—COMPARISON OF OBSERVED AND CALCULATED LEGITIMATE BIRTHS, SCOTLAND.

Census Year.	Observed.	Calculated.	Percentage Error.
1861	97,080	97,044	− 0·04
1871	105,051	104,720	− 0·32
1881	115,687	116,336	+ 0·56
1891	116,339	116,113	− 0·19
1901	123,833	123,655	− 0·14
1911	112,650	113,193	+ 0·48
1921	114,444	102,505	− 10·43
1931	85,559	85,540	− 0·02

It will be seen that, excluding 1921, the error never exceeds 0·6%, and that the average error is 0·25%.

III. COMPARISON OF CALCULATED VALUES OF SPECIFIC FERTILITY RATES WITH VALUES ESTIMATED ON BASIS OF 1931 CENSUS RETURNS.

As mentioned above, the specific fertility rates calculated for England and Wales on the basis of the diagonal law were compared with those estimated by the Registrar-General from the 1931 census returns. As similar estimates had not been made for Scotland the Registrar-General had a special count made on a sample of approximately one-tenth of the census schedules. The results, which have been published in the *Annual Report of the Registrar-General for Scotland, 1937*, are summarised in Table IX, which also contains a statement of the standard errors. These

are calculated from the formula $S.E. = \frac{\text{Rate}}{\sqrt{N}}$, where N is the number of observations from which the rate is derived. We should like to emphasise the importance of some measure being given of the "sampling error" in papers and official reports on vital statistics. It is not uncommon, for example, for birth-rates based on a few score births to be stated to one or two decimal places, and accompanied by comments as to their being greater or smaller than those of the previous week or month, when a statement of the standard error would show at once that no significance whatsoever could be attached to any apparent change.

TABLE IX.—OBSERVED SPECIFIC LEGITIMATE FERTILITY RATES, SCOTLAND, 1931.

Age-group.	All Scotland.		Large Burghs.		Small Burghs.		Landward Areas.	
	Rate.	S.E.	Rate.	S.E.	Rate.	S.E.	Rate.	S.E.
15–	588·2	± 35·9	541·6	± 44·6	619·8	± 96·8	679·6	± 76·5
20–	386·5	± 9·3	388·9	± 12·3	397·6	± 23·9	373·6	± 17·7
25–	256·8	± 5·3	256·9	± 7·0	244·1	± 12·8	262·5	± 10·3
30–	166·1	± 3·9	166·5	± 5·1	134·3	± 8·5	183·7	± 7·8
35–	107·3	± 3·1	108·0	± 4·2	101·4	± 7·5	108·9	± 6·0
40–	35·9	± 1·9	34·1	± 2·5	30·6	± 4·2	42·6	± 3·9
45–	4·3	± 0·7	4·1	± 0·9	3·7	± 1·5	4·8	± 1·3

If the specific legitimate fertility rates shown in Table IX be compared with those calculated in Table VII, Section II, it will be seen that considerable divergences occur. The census estimate gives higher values for the younger, and lower values for the older age-groups. It is of interest to compare the Scottish figures with those of Denmark, which are given in our previous paper. In the latter, the general agreement with the diagonal law is very good except for the last quinquennial period (1931–1935), where the figures show the same type of discrepancy as that found in the case of Scotland, namely, an excess in the observed values for the younger, and a deficiency for the older ages. In the case of the Scottish data, the total legitimate births for all census years from 1861 onwards are in excellent agreement with the numbers calculated from the theoretical fertility rates given in Table VII. It is, therefore, not impossible that, as in the case of Denmark, the predictions for the years other than 1931 are much closer to the actual values than the discrepancy during this last year would suggest.

It seemed desirable to use the specific legitimate fertility rates derived from the census count in order to arrive at some estimate of the specific total fertility rates (total births per 1000 women at various ages). For this purpose it was necessary to make some assumption as to the distribution of illegitimate births according to the age of the mother. We may assume that the percentage fall in the specific illegitimate fertility rates has been independent of age, as has roughly been the case in Sweden. The values of these rates for 1855 may be calculated from the figures of Lewis and Lewis, and are as follows:—

Age-group .	15–	20–	25–	30–	35–	40–	45–
Rate . .	6·4	28·4	30·1	19·8	11·9	3·5	0·5

The numbers of unmarried women of different ages in 1931 are known, and, when the above rates are applied to these numbers, the illegitimate births to be expected number 11,248. The actual number of illegitimate births in 1931 was 6661, so that the specific illegitimate rates for 1855 must be multiplied by $\frac{6,661}{11,248}$ in order to obtain the specific rates for 1931. The illegitimate births to be attributed to unmarried women of various ages are shown in Table X, and, from these and the legitimate births, the total births for mothers of various ages are readily found. The specific total fertility rates are then obtained, and are given in the same table. The "total fertility" (as defined by Kuczynski, 1935, p. 117) according to this assumption is 2369·0 per 1000 women, and the corresponding "gross reproduction rate" is 1·162.

TABLE X.—CALCULATION OF APPROXIMATE SPECIFIC TOTAL FERTILITY RATES, SCOTLAND, 1931.

Age-group.	Illegitimate Births (estimated). (1)	Legitimate Births (derived from Census Count). (2)	Total Births. (3)	Total Women. (4)	Specific Total Fertility Rate. (5)
			(1) + (2)		$\frac{1000 \times (3)}{(4)}$
15–	815	2,940	3,755	220,130	17·1
20–	2,801	19,001	21,802	215,785	101·0
25–	1,812	25,879	27,691	202,487	136·8
30–	753	20,438	21,191	187,248	113·2
35–	367	12,890	13,257	172,262	77·0
40–	98	3,960	4,058	157,769	25·7
45–	14	439	453	150,694	3·0

IV. Local Legitimate Fertility Rates.

A review has recently been carried out by Dr Enid Charles (1938) of the total fertility in the various counties and principal towns of Scotland. She employs as the index of fertility the "gross reproduction rate" of Kuczynski (1935, p. 120), which may be defined as the average number of female children born to a woman in passing through the child-bearing period, it being assumed that she does not die before the period is completed. In a country such as Scotland, before 1938, in which specific fertility rates were not available owing to the deficiency of the information recorded at the registration of a birth, these gross reproduction rates could be obtained only indirectly. It can be shown that the same figures, apart from a common factor, are obtained by the method of regional correction factors applied to the general total fertility rates—that is, the total births per annum per 1000 women aged 15–45. It follows that, if either rate be expressed as a percentage of the corresponding rate for Scotland as a whole, the resulting values are identical.

In order to discover the causes of changes in fertility, it is desirable to focus attention on the fertility of married women, either by suitable modifications of the gross reproduction rates (*cf.* Charles, 1939), or by using the standardised general legitimate fertility rates employed below. The effect of variation in the proportion of women who are married may thus be eliminated.

The Registrar-General has been in the habit of publishing, for the whole of Scotland, and also for various subdivisions, the general legitimate fertility rate, the number of legitimate births per 1000 married women, between the ages of 15 and 45. As a measure of the legitimate fertility, this figure suffers from the disadvantage that it does not make allowance for variations in the age distribution of the women. As will be seen from the figures quoted in Section III, the specific fertility rate differs markedly from age to age. It is necessary, therefore, to make a correction, or adjustment, so as to allow for this effect. Some early calculations with this object in view were made by Newsholme and Stevenson in 1905 and 1906. The method of adjustment used is analogous to that employed by the Registrar-General in correcting the crude death-rates, and is exactly equivalent to the correction of the general total fertility rate alluded to above.

Let us denote by $F_1, F_2, \ldots F_7$, the specific legitimate fertility rates of Scotland as a whole; by F, the general legitimate fertility rate of Scotland as a whole; by f, the general legitimate fertility rate of the Local Area; by $m_1, m_2, \ldots m_7$, the numbers of married women in the

various age-groups in the Local Area; and by m, the number of married women aged 15–45 in the Local Area. If the specific legitimate fertility rates of the Local Area were the same as those of Scotland, the number of legitimate births in this Area would be $m_1F_1 + m_2F_2 + \ldots + m_7F_7$, $= \Sigma m_\theta F_\theta$, and the calculated general legitimate fertility rate for the Local Area would be $\frac{\Sigma m_\theta F_\theta}{m}$, which we denote by f'. Now the application of the standard rates to the age distribution of married women for the whole of Scotland gives the actual general legitimate fertility rate for Scotland, namely, F. The ratio $\frac{F}{f'}$ is then the Local Area correction factor. The standardised local rate is $f \cdot \frac{F}{f'}$, and the ratio of the standardised rate to the observed rate for Scotland is $f \cdot \frac{F}{f'} \cdot \frac{1}{F}, = \frac{f}{f'}$. But this is equal to the ratio of the actual number of births occurring in the Local Area to the calculated number (based on the specific legitimate fertility rates for the whole of Scotland). Thus, if we are interested only in the order of the various districts, it is sufficient to work out the value of this ratio; to obtain the actual standardised rates these ratios must be multiplied by F.

In view of the differences between Large Burghs, Small Burghs, and Landward Areas, brought out in Section III, it is clearly desirable to deal with these three divisions separately. For regional subdivision the counties are obviously the most suitable, as the available data are already, in part, arranged according to these administrative areas. We have then, in all, 90 districts, namely, 33 Landward Areas, 33 groups of Small Burghs, and 24 Large Burghs.

In order to avoid the errors resulting from the use of too small numbers, it was decided to employ as a basis the ten-year period 1926–1935. The total legitimate births for these ten years were calculated from official sources for the 90 districts separately. The distributions of married women used were those given by the census of 1931. For our purpose it was desirable to employ, not the numbers as enumerated, but the numbers permanently resident in the district. A substantial discrepancy between this number and the figure enumerated is occasionally found. In Bute Small Burghs (Rothesay and Millport), for example, the enumerated married women aged 15–45 exceeded the resident by 34·7%. The resident population of married women arranged according to age, for the various districts, has not been published, but, by the courtesy of the Registrar-General, we have been enabled to calculate them from data available in the Register House. The calculation of the actual number of

births in the districts during the decennial period also involved a number of minor adjustments, most of which were occasioned by the changes in classification of Burghs which were introduced in 1931. The results are presented on the basis of the local areas as these were in that year.

The observed and standardised general legitimate fertility rates of the various counties of Scotland are shown in Tables XI and XII respec-

TABLE XI.—OBSERVED GENERAL LEGITIMATE FERTILITY RATES, COUNTIES OF SCOTLAND, 1926–1935.

	Small Burghs.				Landward Areas.			
County.	Rate.	S.E.	Relative Rate: Scotland = 100.	Order.	Rate.	S.E.	Relative Rate: Scotland = 100.	Order.
Aggregate	157·3	± 0·5	92·6		179·9	± 0·4	105·9	
Aberdeen	144·3	± 1·8	84·9	25	190·8	± 1·4	112·3	8
Angus	140·8	± 2·0	82·9	26	167·1	± 2·2	98·3	22
Argyll	171·5	± 3·1	100·9	7	168·1	± 2·4	98·9	21
Ayr	157·5	± 1·3	92·7	16	182·8	± 1·2	107·6	12
Banff	177·1	± 2·4	104·2	3	198·4	± 3·0	116·8	4
Berwick	128·3	± 4·5	75·5	32	138·8	± 2·6	81·7	32
Bute	155·4	± 4·4	91·5	19	144·6	± 5·3	85·1	29
Caithness	203·4	± 4·5	119·7	1	187·3	± 4·0	110·2	10
Clackmannan	148·9	± 2·5	87·6	22	157·8	± 3·7	92·9	25
Dunbarton	159·7	± 2·5	94·0	11	171·3	± 1·8	100·8	17
Dumfries	148·8	± 3·3	87·6	23	169·7	± 2·0	99·9	20
East Lothian	172·8	± 2·8	101·7	5	166·0	± 2·5	97·7	24
Fife	145·7	± 1·2	85·7	24	171·2	± 1·2	100·8	18
Inverness	184·1	± 7·7	108·3	2	205·3	± 2·3	120·8	3
Kincardine	155·1	± 4·7	91·3	20	186·8	± 3·2	109·9	11
Kinross	171·7	± 8·5	101·1	6	153·7	± 5·9	90·5	26
Kirkcudbright	158·6	± 4·1	93·3	13	178·9	± 3·1	105·3	15
Lanark	156·9	± 4·6	92·3	17	192·0	± 0·8	113·0	6
Midlothian	158·8	± 2·0	93·5	12	170·0	± 1·8	100·1	19
Moray	158·1	± 2·8	93·0	14	193·2	± 3·3	113·7	5
Nairn	169·5	± 7·1	99·8	8	210·2	± 7·8	123·7	2
Orkney	138·3	± 5·5	81·4	28	171·4	± 3·5	100·9	16
Peebles	133·4	± 4·1	78·5	30	134·5	± 4·6	79·2	33
Perth	152·6	± 2·8	89·8	21	152·9	± 1·7	90·0	27
Renfrew	166·5	± 1·7	98·0	9	151·6	± 1·7	89·2	28
Ross and Cromarty	155·6	± 4·1	91·6	18	215·0	± 2·5	126·5	1
Roxburgh	136·2	± 2·3	80·2	29	143·3	± 2·8	84·3	31
Selkirk	120·0	± 2·5	70·6	33	143·7	± 6·2	84·6	30
Stirling	163·7	± 2·3	96·3	10	181·9	± 1·5	107·1	13
Sutherland	138·7	± 15·0	81·6	27	191·0	± 4·2	112·4	7
West Lothian	174·0	± 2·2	102·4	4	189·6	± 1·9	111·6	9
Wigtown	157·7	± 3·8	92·8	15	181·3	± 3·2	106·7	14
Zetland	129·5	± 6·0	76·2	31	166·5	± 3·7	98·0	23

tively. The corresponding rates for the 24 Large Burghs are presented in Table XIII. Each table contains not only the general fertility rate, but also its value in terms of Scotland as 100, and its standard error. The geographical distribution of the standardised rates for the Landward Areas and the Small Burghs is represented pictorially in figs. 1 and 2.

TABLE XII.—STANDARDISED GENERAL LEGITIMATE FERTILITY RATES, COUNTIES OF SCOTLAND, 1926–1935.

County.	Small Burghs.				Landward Areas.			
	Rate.	S.E.	Relative Rate: Scotland = 100.	Order.	Rate.	S.E.	Relative Rate: Scotland = 100.	Order.
Aggregate	160	± 0·5	94·2		182	± 0·4	107·2	
Aberdeen	138	± 1·8	81·3	30	183	± 1·4	107·7	13
Angus	149	± 2·2	87·6	27	173	± 2·2	101·8	19
Argyll	188	± 3·3	110·4	2	188	± 2·7	110·8	10
Ayr	160	± 1·3	93·9	17	178	± 1·2	104·7	17
Banff	177	± 2·4	104·1	6	199	± 3·0	117·2	5
Berwick	133	± 4·7	78·3	32	147	± 2·8	86·5	33
Bute	165	± 4·7	97·0	13	160	± 5·8	93·9	28
Caithness	194	± 4·3	114·2	1	194	± 4·2	114·3	7
Clackmannan	150	± 2·6	88·5	26	157	± 3·7	92·5	29
Dunbarton	170	± 2·6	100·0	10	179	± 1·9	105·4	16
Dumfries	153	± 3·4	89·9	23	173	± 2·0	101·6	20
East Lothian	169	± 2·7	99·6	11	166	± 2·5	97·7	26
Fife	145	± 1·2	85·4	28	168	± 1·2	98·6	24
Inverness	186	± 7·7	109·5	3	235	± 2·7	138·3	2
Kincardine	157	± 4·8	92·4	21	186	± 3·2	109·6	11
Kinross	178	± 8·8	104·6	5	168	± 6·5	99·0	25
Kirkcudbright	160	± 4·1	94·3	18	181	± 3·1	106·6	14
Lanark	158	± 4·6	92·7	20	191	± 0·8	112·3	9
Midlothian	162	± 2·0	95·2	15	169	± 1·8	99·6	23
Moray	163	± 2·9	95·9	14	193	± 3·3	113·6	8
Nairn	180	± 7·5	105·7	4	223	± 8·3	131·4	3
Orkney	154	± 6·1	90·8	22	172	± 3·5	101·1	21
Peebles	136	± 4·2	80·2	31	150	± 5·1	88·4	32
Perth	160	± 2·9	93·9	19	161	± 1·8	94·7	27
Renfrew	175	± 1·8	103·0	7	171	± 2·0	100·5	22
Ross and Cromarty	174	± 4·6	102·2	8	255	± 2·9	149·9	1
Roxburgh	141	± 2·4	83·1	29	154	± 3·0	90·8	31
Selkirk	128	± 2·7	75·6	33	157	± 6·7	92·5	30
Stirling	162	± 2·3	95·3	16	176	± 1·4	103·7	18
Sutherland	168	± 18·1	98·9	12	205	± 4·5	120·6	4
West Lothian	172	± 2·1	101·0	9	185	± 1·9	108·8	12
Wigtown	153	± 3·7	90·2	24	181	± 3·2	106·7	15
Zetland	152	± 7·1	89·4	25	196	± 4·3	115·6	6

TABLE XIII.—OBSERVED AND STANDARDISED GENERAL LEGITIMATE FERTILITY RATES, LARGE BURGHS OF SCOTLAND, 1926–1935.

Large Burgh.	Observed. Rate.	Observed. S.E.	Observed. Relative Rate: Scotland = 100.	Observed. Order.	Standardised. Rate.	Standardised. S.E.	Standardised. Relative Rate: Scotland = 100.	Standardised. Order.
Aggregate	168·2	± 0·2	99·0		167	± 0·2	98·0	
Glasgow	173·9	± 0·4	102·3	8	172	± 0·4	101·4	8
Edinburgh	146·8	± 0·6	86·4	22	149	± 0·6	87·5	21
Dundee	166·5	± 0·9	98·0	12	164	± 0·9	96·3	14
Aberdeen	159·0	± 0·9	93·6	17	153	± 0·9	90·2	18
Paisley	172·2	± 1·3	101·3	9	172	± 1·3	101·5	9
Greenock	195·5	± 1·5	115·1	5	189	± 1·5	111·2	5
Motherwell and Wishaw	183·6	± 1·6	108·1	7	179	± 1·5	105·6	7
Clydebank	168·9	± 1·8	99·4	11	170	± 1·8	100·0	10
Kirkcaldy	148·7	± 1·7	87·5	21	149	± 1·7	87·9	22
Coatbridge	212·1	± 2·1	124·8	2	204	± 2·0	119·9	2
Kilmarnock	160·4	± 1·9	94·4	16	156	± 1·9	92·1	16
Hamilton	199·7	± 2·2	117·5	3	187	± 2·0	110·1	6
Ayr	166·3	± 2·0	97·9	13	160	± 2·0	94·3	15
Falkirk	159·0	± 1·9	93·6	18	152	± 1·9	89·2	20
Dunfermline	139·3	± 1·8	82·0	24	147	± 1·9	86·3	24
Perth	145·0	± 2·0	85·3	23	148	± 2·0	87·3	23
Airdrie	197·3	± 2·6	116·1	4	192	± 2·5	112·9	3
Rutherglen	150·9	± 2·4	88·8	20	154	± 2·4	90·6	17
Dumfries	155·7	± 2·5	91·6	19	152	± 2·4	89·2	19
Stirling	172·1	± 2·7	101·3	10	165	± 2·5	96·9	12
Inverness	163·3	± 2·7	96·1	14	165	± 2·7	97·1	11
Dumbarton	183·7	± 2·9	108·1	6	189	± 2·9	111·0	4
Port-Glasgow	220·3	± 3·2	129·7	1	211	± 3·1	124·4	1
Arbroath	163·2	± 3·1	96·0	15	164	± 3·1	96·7	13

It will be seen that the effect of standardisation is, in some instances, considerable. Thus, in the case of Small Burghs, Argyll, which in the unstandardised list is seventh in rank with a value of 171·5, rises to second place (value 188); Ross and Cromarty (unstandardised 155·6, standardised 174) rises from eighteenth to eighth place; whilst Orkney (unstandardised 138·3, standardised 154) rises from twenty-eighth to twenty-second place. Similar changes are shown by the Landward Areas: Argyll, for instance, rises from twenty-first to tenth place. In the case of the Large Burghs, the change of order on standardisation is not so marked. It is of interest to notice that the four Border counties,

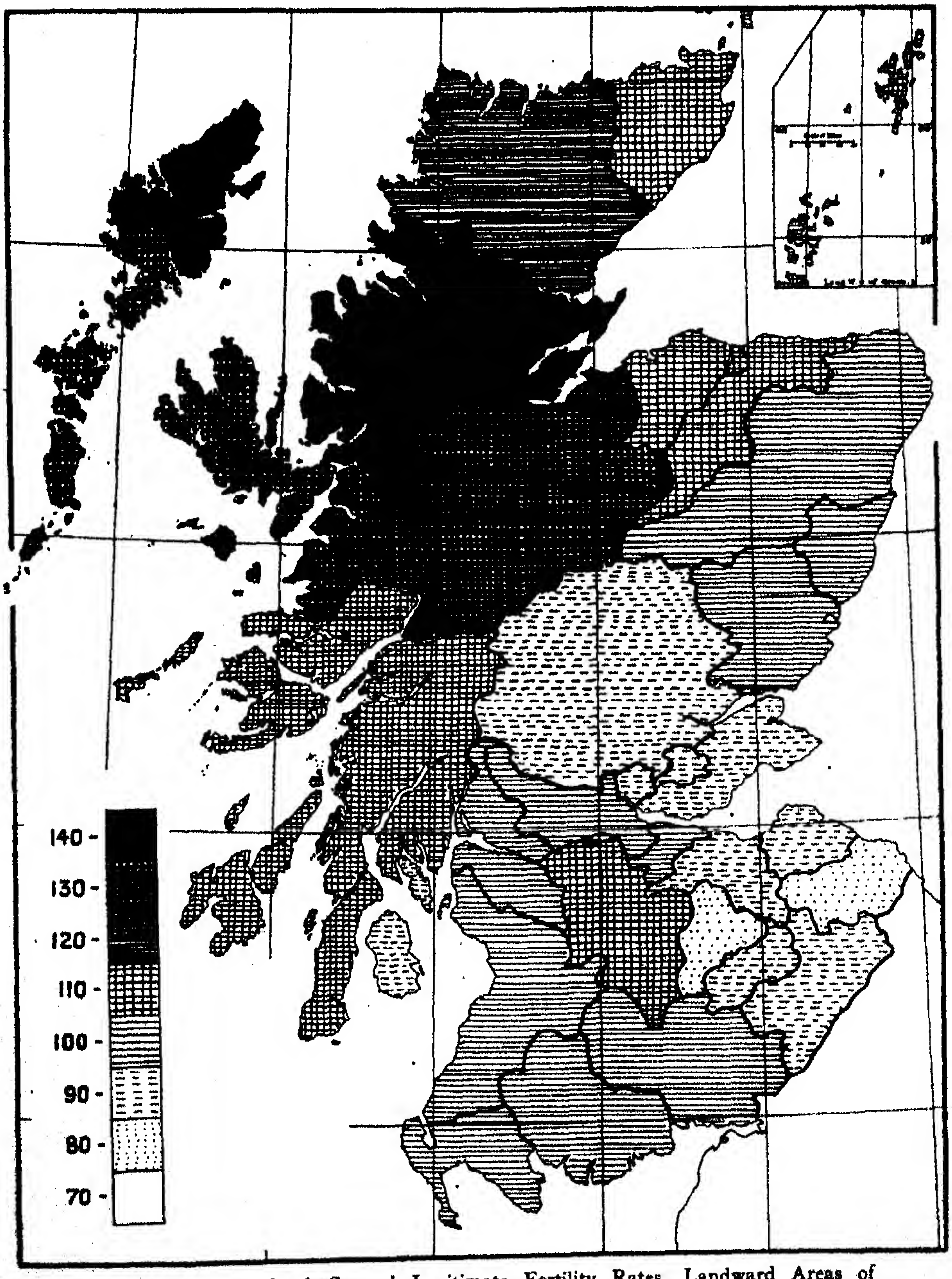


FIG. 1.—Standardised General Legitimate Fertility Rates, Landward Areas of Scotland, 1926–1935. (The general legitimate fertility rate of Scotland, 1926–1935, is taken as 100.)

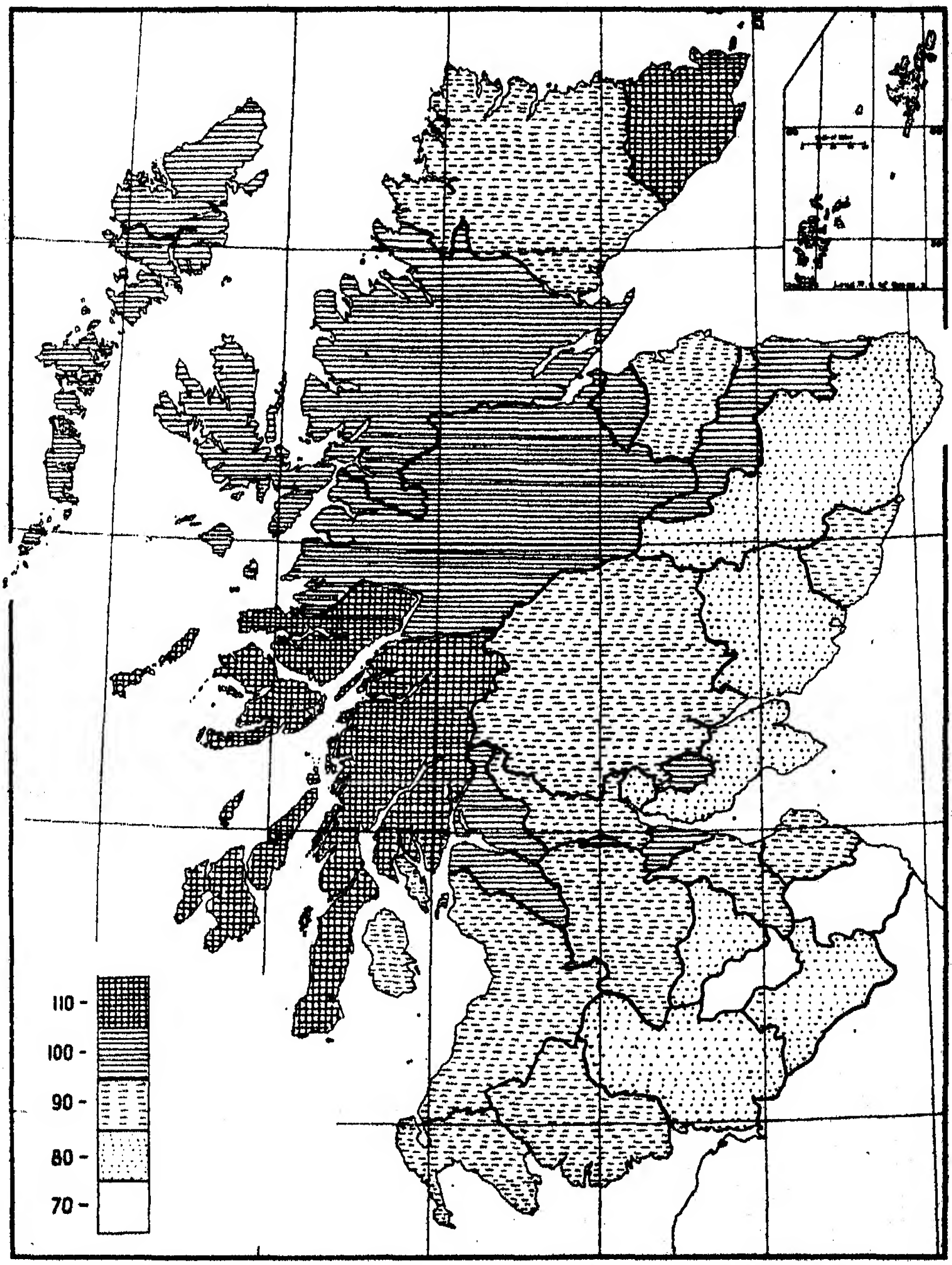


Fig. 2.—Standardised General Legitimate Fertility Rates, Small Burghs of Scotland, 1926–1935. (The general legitimate fertility rate of Scotland, 1926–1935, is taken as 100.)

Berwick, Roxburgh, Peebles, and Selkirk, are at the foot of all the tables, Landward or Small Burghs, standardised or unstandardised.

When the rates are expressed in terms of that of Scotland as a whole, there is a difference, between standardised and unstandardised, of 5–10% in 8 out of the 33 counties, in the case of the Small Burghs, and of over 10% in 3. For the Landward Areas, the corresponding figures are 7 and 5 respectively. Two Large Burghs show changes of over 5%.

The standardised rates for the aggregates of Small Burghs, Large Burghs, and Landward Areas are 160, 167, and 182 respectively, that of the whole of Scotland being 169·9. The lowest rates are found in the Small Burghs of Selkirk (128) and Berwick (133), whilst the highest are in the Landward Areas of Ross and Cromarty (255) and Inverness (235). The Large Burghs also exhibit a very considerable range, from Dunfermline with 147 to Port-Glasgow with 211.

Two main points emerge from Tables XI and XII, and figs. 1 and 2. First, fertility is greater in the Landward districts than in the Small Burghs, not only throughout Scotland as a whole, but also in almost every single county. Where the order is disturbed, as in the case of East Lothian, the total number of births involved is not large enough for the results to be significant. Secondly, the Landward Areas show a fairly regular sequence in their fertilities. The highest value is found in Ross and Cromarty. On moving northwards, or southwards and eastwards, from this point, we observe progressively lower values. Thus, the lowest fertilities are in the south and east, in Roxburgh, Peebles, Selkirk, and Berwick. A parallel tendency is noticeable in the case of the Small Burghs, but the regularity is not so great. The Large Burghs are not sufficiently widely distributed throughout Scotland to enable any analogous conclusions to be drawn. But it is of interest to note that the ten Burghs with the largest fertility rates are all at the western end of the industrial belt, whilst the seven of lowest fertility are either on the eastern seaboard or in the extreme south (Dumfries).

The case of Ross and Cromarty (Landward Area) is of special interest. The fertility rate, either crude or standardised, is the highest of all, and it is also peculiar in that the effect of the correction is greater than for any other. This large correction is a consequence of the fact that the average age of the married women in this county is abnormally high, which, in turn, is, at least partly, due to the circumstance that the average age of marriage is unusually great, a peculiarity pointed out by Dr Charles in her discussion of Scottish total fertility. Now this high average age at marriage, which is characteristic of all the crofting counties, means that women of age, say 40, will, on the average, have been married a

considerably shorter time than their contemporaries elsewhere in Scotland. It might be anticipated that the fertility of women depends not only on their age, but also on the duration of their marriage. We have been very kindly supplied by the Statistical Institute of the University of Lund (*cf.* Wicksell and Quensel, 1938) with fertility rates which have been calculated for Swedish women, classified in respect both of age and duration of marriage. Examination of these statistics shows that, in relation to fertility, the duration of marriage is even more important than age. A consequence of the great importance of duration of marriage is that, other things being equal, these Ross and Cromarty women, simply because of their later marriages, would appear to be unusually fertile. This would be the case even though their real fertility, the fertility rates considered as functions of both age and duration of marriage, was no greater than that of the rest of Scotland. This does not, of course, alter the fact that the age specific fertility rates of these women are abnormally high, but it does suggest an explanation for this peculiarity.

This example illustrates the disadvantage of analysing fertility in terms of the age of women alone, and emphasises the importance of the official statistics being so presented that the effect of duration of marriage can also be taken into account. At the present time, as far as we know, this can be done only in the case of the special Swedish statistics referred to above.

V. Discussion.

Amongst the more noteworthy features revealed by the present investigation is the relatively low fertility of the Small Burghs as compared with the Landward Areas, not only in Scotland as a whole, but also in the great majority of the counties taken separately. Of the four Cities, Glasgow and Dundee have fertilities of intermediate value, but Edinburgh and Aberdeen are less fertile than the average Small Burgh. The other Large Burghs show considerable variation, those in the west being, in general, much more fertile than those in the east. There is no striking deficiency in the fertility of the large towns as compared with that of the rural areas, such as is a marked characteristic of many industrialised countries—for example, the United States or Germany. In this respect Scotland resembles England and Wales. There is thus little evidence that the decrease of fertility in Scotland is in any direct way to be associated with the progressive urbanisation to which the country has been subjected. Indeed, it would seem to be a reasonable expectation that, if the Large Cities and other Large Burghs were replaced by numerous Small Burghs of semi-rural aspect, the fertility would be even lower than it is at the present time.

The results presented in this paper may be regarded as complementary to those of Dr Charles. This author has concentrated attention on total fertility, and has made it clear that, in relation to this, the degree of nuptiality is a very important factor, fertility being low when there is a large percentage of unmarried women in the community. This would seem to be an important factor in reducing the total fertility in textile manufacturing districts, where many women are employed in mills, and where, in consequence, there is an unusually high proportion of single women. It is interesting to notice that, in the case of legitimate fertility, this is low in some textile districts, such as Selkirk, but high in others—for example, Paisley.

The results here reported serve to give a general indication of the state of Scottish fertility in 1931. Within the next year or two the results of the new Population (Statistics) Act will become available, and this will afford direct and reliable information on many points discussed above. However, in the case of most areas, several years must elapse before the numbers of recorded births are sufficiently large to yield estimates of local fertility of reasonable precision. Under these circumstances, it is considered that even approximate results for 1931 are of some value, both for the immediate light they shed on Scottish fertility, and as a basis of comparison when future results become available.

In conclusion, we wish to draw attention to the desirability of analysing birth returns in respect of duration of marriage, as well as of mother's age. Under the new Population (Statistics) Act, the registration of births in Scotland involves the statement of mother's age as well as of the date of marriage. It should, therefore, be possible to obtain further information as to the relative importance and mutual effect of these two factors.

Summary.

1. When specific legitimate fertility rates are calculated for Norway on the basis of the assumptions, including the validity of the "diagonal law" previously used in calculating the rates for England and Wales, good agreement is found with the observed specific fertility rates, which, in this country, are available only for certain periods.

2. Specific legitimate fertility rates have been calculated for Scotland on the basis of similar assumptions. The total legitimate births calculated from these rates are in good agreement with the numbers observed, but, for the last period, the calculated values of the legitimate fertility rates show serious discrepancies from the values deduced from the 1931 census returns. The deviations are similar to those found in the case of Denmark for the period 1931–1935.

3. The specific legitimate fertility rates for Scotland 1931 have been employed to standardise the general legitimate fertility rates for separate areas, viz., the Large Burghs, and, for each county, the Small Burghs and Landward Areas.

4. In general, the Landward Areas show the greatest legitimate fertility, and the Small Burghs the least, whilst the Large Burghs are intermediate. In the Landward Areas legitimate fertility is greatest in Ross and Cromarty, and decreases fairly regularly southwards and eastwards, as well as to the north. A somewhat parallel trend is observed for the Small Burghs, though, in this case, the results are somewhat less regular. The lowest fertilities are found in the Small Burghs to the south-east, *e.g.*, in Berwickshire.

We wish to express our thanks to the Medical Research Council for a grant to one of us (R. S. B.), and to the Director of the Norwegian Central Bureau of Statistics, Oslo, for data used in Section II. We are also indebted to Mr A. McKinlay and Mr Graham of the Registrar-General's Office, Edinburgh, for much valuable assistance.

REFERENCES TO LITERATURE.

BARCLAY, R. S., and KERMACK, W. O., 1938. *Proc. Roy. Soc. Edin.*, vol. lviii, pp. 55–72.

CHARLES, E., 1938. *Trans. Roy. Soc. Edin.*, vol. lix, pp. 371–83.

——, 1939. *Trans. Roy. Soc. Edin.*, vol. lix, pp. 673–686.

Eighty-Third Annual Report of the Registrar-General for Scotland, 1937, pp. li–liv.

KUCZYNSKI, R. R., 1935. *The Measurement of Population Growth.*

LEWIS, C. J., and LEWIS, J. N., 1906. *Natality and Fecundity*, Table XIV, p. 29.

NEWSHOLME, A., and STEVENSON, T. H. C., 1905. *Journ. Hyg.*, vol. v, pp. 175–84 and 304–10.

——, and ——, 1906. *Journ. Roy. Stat. Soc.*, vol. lxix, pp. 34–87.

Norges Offisielle Statistikk, Folkemengdens Bevegelse, 1921–1932, Table k, p. 24.

WICKSELL, S. D., and QUENSEL, C.-E., 1938. *Statens Off. Utr.*, 1938: 24 (Sweden).

(*Issued separately April* 12, 1939.)

VIII.—Lunar Atmospheric Pressure Variations at Glasgow.

By **R. A. Robb,** M.A., D.Sc., and **T. R. Tannahill,** M.A., B.Sc.

(MS. received September 10, 1938. Revised MS. received January 3, 1939. Read February 6, 1939.)

1. IN a previous communication (Robb and Tannahill, 1935) the lunar atmospheric pressure inequalities at Glasgow for days whose range of pressure was less than or equal to 0·1 in. of mercury were investigated. It was found that a large first harmonic was present and that the second harmonic or semi-diurnal component had a phase angle of 285° in contradiction to the phase angle of 90° as predicted by tidal theory.

In a paper by Chapman (1936) reasons were given for believing that these inequalities are substantially not of lunar origin at all, but that, owing to the method of tabulation adopted in the Glasgow reductions, the "convexity" effect, first discovered by Bartels (1927, p. 26), was not eliminated as it had been in Chapman's Greenwich reductions (1918, p. 271). The convexity effect mentioned is described by Chapman and Austin (1934) as follows:—

> ". . . barometrically quiet days tend to occur at times of high pressure, and therefore near the maxima of pressure; consequently on such days the form of the barometric curve is convex (d^2p/dt^2 negative). . . ."

It can be represented by a parabolic arc with its maximum near the centre of a quiet day, superposed on the usual periodic variations. That is, the lunar pressure inequalities for any one day are given by

$$y_x = p_x + \text{periodic terms},$$

where p_x is the ordinate of the parabola at hour x. It should be emphasised that p_x is non-periodic and must be eliminated before the true periodic effects can be found. For lunar days starting at upper transit the convexity is a maximum near lower transit, while for lunar days starting at lower transit the convexity is a maximum near upper transit.

2. For the discussion of lunar inequalities four methods of selecting

days are considered in this paper. They are denoted by α, β, γ, and ϵ, as follows:—

Type of Day.	Range of Pressure in Inches.	Beginning.	End.	Length.
α	0·1	23^h solar time.	23^h solar time.	25 solar hours.
β	0·1	Upper lunar transit.	Upper lunar transit.	One lunar day of 25 solar hours.
γ	0·1	Lower lunar transit.	Lower lunar transit.	One lunar day of 25 solar hours.
ϵ	α-days lying completely within β-days.			

3. In our previous note (1935, p. 91) we discussed the lunar inequalities derived from lunar days, measured from upper transit to upper transit, whose range of pressure over 25 solar hours did not exceed 0·1 in. These days are called β-days in this paper (§ 2). As pointed out by Chapman (1936, p. 1), the results obtained did not, however, give a true value of the lunar variation, in so far as the convexity effect had not been eliminated. From an examination of the results obtained by one of the authors in 1926, by the method of selection of days used for the Greenwich data (Chapman, 1918), Chapman came to the conclusion that the true lunar (semi-diurnal) tide at Glasgow is very small with an approximate amplitude of 5 microbars and a phase angle between 50° and 90°. A diurnal component was ignored as being accidental. He verified this result from a discussion of the "β-day" inequalities on the assumption that the entire diurnal component was due to convexity. In a criticism of the method of selection of lunar days Chapman made the suggestion that if tabulations of lunar days had been made starting at lower transit instead of at upper transit as in our investigation, substantially the same diurnal component, with its maximum at the centre of the lunar day, that is, at upper transit, would have been obtained. This statement is based on the assumption that there is no periodic lunar diurnal component.

We have carried out the work necessary to examine this suggestion. The inequalities obtained from lunar days starting at lower lunar transit, that is, γ-days, are given in columns (2) to (5) of Table I. It should be pointed out that as no lower transit times are tabulated in the *Nautical Almanac* for the period 1868–1882, interpolated values of the inequalities are used for this period. Columns (6) to (9) of Table I give the corresponding inequalities for β-days, viz. days starting at upper transit. These latter inequalities differ from those in Table II of our previous

note (1935, p. 94) in so far as the correction for solar variation is derived from the solar inequalities obtained from γ-days, and, secondly, a correction for (linear) non-cyclic change is applied. This latter correction is obtained by tabulating the 26th hour of the lunar day and taking its difference from the initial hour. The lower transit inequalities were similarly corrected.

TABLE I.—LUNAR INEQUALITIES FROM [A] γ-DAYS, [B] β-DAYS.

Unit ·0001 millibar.

Hour.	[A] Lower Transit.				[B] Upper Transit.			
	Winter.	Equinoxes.	Summer.	Total.	Winter.	Equinoxes.	Summer.	Total.
0½	−727	−448	−346	−459	−1477	−1256	−809	−1096
1½	−388	−335	−307	−331	−1174	−964	−624	−851
2½	−178	−263	−295	−257	−820	−693	−489	−625
3½	−19	−272	−216	−191	−601	−458	−307	−418
4½	+27	−152	−237	−151	−328	−223	−185	−228
5½	+111	−145	−194	−107	−184	−96	−105	−119
6½	+72	−111	−196	−110	−89	+105	−14	+8
7½	+102	+14	−125	−26	−49	+320	+113	+144
8½	+153	−27	−70	−7	+110	+487	+198	+272
9½	+139	+12	+38	+48	+123	+553	+347	+365
10½	+124	−12	+107	+82	+423	+631	+444	+499
11½	−7	+39	+184	+97	+516	+722	+502	+575
12½	+44	+113	+180	+129	+679	+740	+553	+640
13½	+137	+163	+204	+176	+774	+670	+408	+570
14½	+232	+153	+202	+200	+758	+644	+389	+548
15½	+240	+97	+226	+185	+851	+508	+364	+510
16½	+162	+60	+208	+162	+785	+426	+360	+473
17½	+249	+92	+198	+171	+680	+309	+258	+364
18½	+165	+124	+212	+174	+622	+164	+154	+260
19½	+125	+182	+145	+151	+376	+100	+17	+121
20½	+81	+250	+88	+150	+266	−83	−78	−6
21½	−49	+215	+75	+88	−98	−327	−195	−217
22½	−175	+173	+97	+66	−354	−548	−285	−384
23½	−238	+105	−28	−39	−716	−768	−422	−596
24½	−380	−16	−147	−172	−1066	−960	−584	−807
No. of days.	927	1355	2005	4287	921	1370	1999	4290

The solar variation used in correcting these lunar inequalities for the slightly irregular distribution of transit hours was derived by transposition of γ-days into solar time. Columns (2) and (5) of Table II show the amount of the correction for each lunar hour. Columns (3) and (6) show the amount of this correction using the normal solar variation

TABLE II.—CORRECTIONS APPLIED TO γ-DAY AND β-DAY LUNAR INEQUALITIES (TOTAL DATA).

Unit ·0001 millibar.

(Subtracted from uncorrected inequalities.)

Hour.	Lower Transit.			Upper Transit.		
	From γ-day S.V.	From normal S.V.	From extra S.V.	From γ-day S.V.	From normal S.V.	From extra S.V.
0½	+17	+14	+3	+50	+42	+8
1½	+8	0	+8	+20	+15	+5
2½	−14	−19	+5	−15	−15	0
3½	−31	−34	+3	−31	−32	+1
4½	−33	−37	+4	−44	−38	−6
5½	−36	−38	+2	−35	−31	−4
6½	−23	−28	+5	−36	−29	−7
7½	−13	−15	+2	−31	−23	−8
8½	+15	+14	+1	−26	−16	−10
9½	+40	+44	−4	−9	+4	−13
10½	+68	+68	0	+10	+21	−11
11½	+59	+64	−5	+30	+37	−7
12½	+72	+74	−2	+26	+29	−3
13½	+41	+42	−1	+17	+19	−2
14½	+18	+21	−3	−11	−10	−1
15½	−5	−4	−1	−21	−20	−1
16½	−27	−21	−6	−32	−37	+5
17½	−37	−35	−2	−33	−39	+6
18½	−48	−44	−4	−27	−34	+7
19½	−43	−42	−1	−5	−14	+9
20½	−35	−31	−4	+13	+4	+9
21½	−16	−15	−1	+48	+40	+8
22½	−1	+3	−4	+70	+63	+7
23½	+18	+16	+2	+67	+64	+3
24½	+17	+14	+3	+50	+42	+8

derived from 45 years data (Becker, 1925). Columns (4) and (7) show the differences (2)–(3) and (5)–(6). This difference is due to the presence of an additional diurnal component in the γ-day solar variation. Its effect is negligible, the maximum correction applied being 13 units, although the extra diurnal component which necessitates the correction has an amplitude of about 950 units. It is evident that on no reasonable assumption regarding the uncertainty of the solar variation can the difference of the upper and lower transit inequalities be explained as being due to this cause.

4. Fig. 1 shows the comparison of the β-day and γ-day inequalities for 1868–1912. It will be seen that, contrary to Chapman's prediction,

the two sets of inequalities do not agree. An examination of Table I shows a similar divergence for each sub-group, indicating the presence of a periodic lunar diurnal component. Thus there are in these inequalities two separate effects: firstly, a convexity effect, as pointed out by Chapman; secondly, a periodic diurnal effect which, with its maximum near lower transit, partially neutralises the convexity effect in the lower

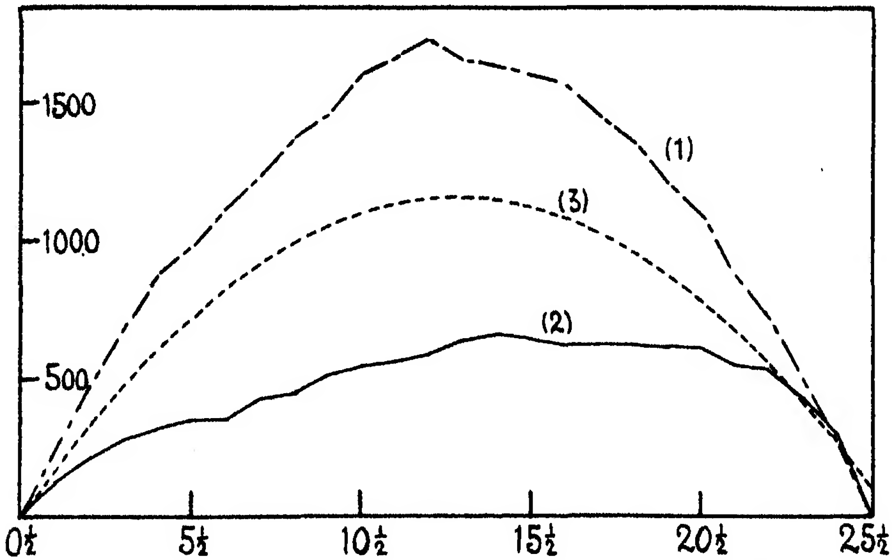


FIG. 1.—Curves (1) and (2) are the β-day and γ-day inequalities, starting at upper and lower transit respectively. Curve (3) is the calculated 25-hour convexity. The unit is ·0001 millibar.

transit inequalities. Convexity effect alone cannot therefore explain the upper transit inequalities.

The existence of such a diurnal component is in contradiction to tidal theory. Nevertheless it is evident that, in days whose daily range of pressure does not exceed 0·1 in., such a component does exist at Glasgow, even when the convexity has been removed.

5. We now require to separate these two effects—the periodic, that is the diurnal component, and the non-periodic, that is the convexity. Two methods will be considered. The first is that adopted by Chapman for the Greenwich data, and consists in the transposition of quiet solar days into lunar time (Chapman, 1936, p. 4). In describing this method as applied to the Greenwich data, Chapman and Austin (1934, p. 25) state:

> " . . . the lunar daily rows there consisted of 25 hourly values from one solar day (midnight to midnight), with the part of the sequence which occurred before lunar transit on that day transposed

so as to succeed the part which followed the transit. This transposition would completely eliminate the convexity if the latter were the same on every day, and if the lunar transit times were uniformly distributed throughout the days used. . . ."

The resultant inequalities give the true lunar effect. This method was adopted by one of the authors in 1926, and the results (for the semi-diurnal component only) have been quoted by Chapman (1936, p. 4).

The quiet solar days used are called α-days in this paper (§ 2). The inequalities found in the present investigation differ from those obtained in 1926 in so far as the following additional corrections have been made:

(1) Linear "non-cyclic change" was removed. This was obtained from the difference of the 23rd solar hour and the 24th solar hour of the following day. It should be mentioned that in correcting for non-cyclic change in this way the effect of the break introduced by transposition is removed.

(2) Owing to the slightly irregular distribution of transit hours the solar variation and convexity are not completely eliminated. In correcting for the residue of these effects remaining after transposition we have used the solar inequalities derived from the actual data. The corrections applied are of the same order of magnitude as those given in Table II.

In Table III [A] the harmonic analysis of these transposed α-day inequalities is shown. It is assumed that the inequalities can be represented by the series

$$\sum_{r=1} c_r \sin (r\theta + \alpha_r),$$

θ being measured from upper lunar transit. A correction $-7^\circ\cdot 5r$ has been applied to each phase angle α_r to allow for the first entry in the lunar tabulations corresponding to $0\frac{1}{2}$ hour lunar time and not to lunar transit. This table shows satisfactory agreement between the various sub-groups of periods and seasons. In particular we may note that the second harmonic appears to be significantly different from zero. An estimate of probable error has been given on page 95 of our previous note. The grouping according to lunar distance will not be considered in this paper.

6. A second method of separating the periodic lunar components from the non-periodic convexity is by the combination of the β-day and γ-day inequalities. The convexity effect in each case is a maximum near the centre of the selected day. That is, for the γ-days it is a maximum near upper transit, while for the β-days it is a maximum near lower transit. Regarding the periodic effect, the odd harmonics will be of the same amplitude but of opposite phase in the two cases, while the even harmonics will agree in amplitude and in phase. Thus, neglecting the

TABLE III.—HARMONIC ANALYSIS OF LUNAR INEQUALITIES FROM [A] TRANSPOSED α-DAYS, [B] COMBINED β- AND γ-DAYS.

Unit ·0001 millibar.

	Periods.			Seasons.			Semi-diameters.			Total.
	1868–1882.	1883–1897.	1898–1912.	Winter.	Equinoxes.	Summer.	$<14'\cdot99$.	15′·00 to 15′·99.	$>16'\cdot00$.	
	[A]									
c_1	122	128	90	108	184	77	186	106	262	99
α_1	279°	263°	336°	256°	321°	261°	40°	294°	251°	290°
c_2	63	105	57	141	35	51	157	72	116	66
α_2	14°	37°	89°	9°	91°	55°	27°	344°	112°	44°
No. of days.	1358	1506	1494	962	1383	2013	1086	1883	1389	4358
	[B]									
c_1	449	86	255	322	363	183	197	201	422	255
α_1	276°	246°	276°	240°	280°	288°	306°	276°	256°	272°
c_2	51	48	56	131	24	41	73	18	12	31
α_2	325°	22°	254°	307°	180°	45°	308°	49°	132°	338°
No. of days β	1347	1492	1451	921	1370	1999	1106	1850	1334	4290
No. of days γ	1326	1488	1473	927	1355	2005	1078	1856	1353	4287

third and higher harmonics, we have for the β-day inequalities

$$y_{x,u} = p_x + c_1 \sin (x + \alpha_1) + c_2 \sin (2x + \alpha_2),$$

where $y_{x,u}$ is the ordinate at hour x, and p_x the convexity at hour x. For the γ-day inequalities we have

$$y_{x,l} = p_x - c_1 \sin (x + \alpha_1) + c_2 \sin (2x + \alpha_2).$$

In both cases x is measured from the beginning of the selected day. Hence by addition and subtraction of these equations we have

$$y_{x,u} + y_{x,l} = 2p_x + 2c_2 \sin (2x + \alpha_2) \quad . \quad . \quad . \quad . \quad (1)$$

and

$$y_{x,u} - y_{x,l} = 2c_1 \sin (x + \alpha_1). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We assume that the convexity can be represented by a parabola, that is,

$$p_x = ax^2 + bx + c,$$

or, if this is harmonically analysed, by

$$p_x = a_1' \sin (x + \beta_1) + \frac{a_1'}{4} \sin (2x + 2\beta_1 + 90^\circ) \qquad . \qquad . \qquad (3)$$

neglecting harmonics higher than the second.

A harmonic analysis of equation (1) yields a first harmonic which is entirely due to convexity, and hence by equation (3) its second harmonic can be obtained. We can now easily obtain from equation (1) the real second harmonic of the lunar effect. Reference to Table III [B] shows that this lunar second harmonic is too small and irregular to be of significance. Equation (1) may therefore be considered to give the convexity effect alone. The 25-hour convexity derived by fitting a parabola by the method of least squares to equation (1) was calculated, and its magnitude from the total data is compared with the β-day and γ-day inequalities in fig. 1.

A harmonic analysis of equation (2) yields the periodic lunar first harmonic, the results being shown in Table III [B]. The agreement in phase among the various groups is satisfactory, showing a definite maximum at lower transit. Comparing with the analysis of the transposed α-day inequalities, we see that the amplitude of the first harmonic has now been substantially increased, the phase remaining approximately the same. In the transposed α-day inequalities, however, we have already noted the probable significance of the second harmonic, whereas in the present case it does not appear to be significant.

7. To investigate further the nature of these discrepancies we have chosen solar days from the quietest parts of the α-day records. This was done by selecting those α-days which lie completely within β-days. These are called ϵ-days. Transposing these days into lunar time, as in § 5, we find inequalities which are analysed in Table IV. In correcting these inequalities for the residue of solar variation and convexity which remains, after transposition, owing to the slightly irregular distribution of transit hours, we have used solar inequalities derived from the ϵ-days themselves.

Comparing Tables III [A] and IV, we see that the effect of selecting the quietest of the α-days is to increase substantially the amplitude of the diurnal component, the phases remaining approximately constant. Thus with increasing quietness of day we find an increase in the amplitude of the first harmonic. Again, comparing the ϵ-day results with those derived from the combined β- and γ-days we see that the diurnal components are in good agreement as regards amplitude and phase. The inference is that the effect of transposing (solar) α-days for

TABLE IV.—HARMONIC ANALYSIS OF LUNAR INEQUALITIES FROM TRANSPOSED ϵ-DAYS.

Unit ·0001 millibar.

	Winter.	Equinoxes.	Summer.	Total.
c_1	380	337	119	228
a_1	271°	282°	270°	275°
c_2	183	69	32	45
a_2	9°	262°	315°	324°
No. of days	302	542	907	1751

lunar purposes is to create "lunar days" which are in effect rougher than the β- or γ-days.

8. We have thus shown the existence of a lunar diurnal component whose amplitude increases with the "quietness" of the day, the phase remaining constant. Associated with this first harmonic there is probably a second harmonic whose effect, as can be seen by a comparison of Tables III [A] and III [B], is partially to neutralise the normal lunar tide.

It is possible, however, with the available data, to obtain a type of day in which these additional harmonics are practically absent. We have found that the first harmonic of the 1751 ϵ-days is 228 sin $(\theta + 275°)$ and hence these days have a total first harmonic 399228 sin $(\theta + 275°)$. In the same way, the 4358 a-days have a total first harmonic 431442 sin $(\theta + 272°)$. Thus the total first harmonic for the 2607 a-days which remain after the ϵ-days have been removed is 112101 sin $(\theta + 356°)$, giving a diurnal component of 43 sin $(\theta + 356°)$. The diurnal effect is therefore practically absent in these days, and we may assume that the additional second harmonic is also absent. That is, the abnormal lunar harmonics observed are confined to the quietest days of the record. Thus, in deriving the normal lunar tide, such days must be entirely excluded.

Applying a similar process to the second harmonics we find the results given in Table V, in which the second harmonics may be taken to represent the normal lunar tide at Glasgow. The second harmonic for the complete period 1868–1912 is

$$0{\cdot}0110 \sin (2\theta + 60°) \text{ millibar,}$$

and is quite consistent in the seasonal results. The corresponding results

for Greenwich and Hongkong, for example, are 0·0120 sin $(2\theta + 114°)$ and 0·060 sin $(2\theta + 60°)$ respectively.

TABLE V.—PROVISIONAL HARMONIC COEFFICIENTS OF THE LUNAR TIDE.

Unit ·0001 millibar.

	Winter.	Equinoxes.	Summer.	Total.
c_1	47	193	46	43
α_1	153°	7°	243°	356°
c_2	123	102	100	110
α_2	8°	87°	69°	60°

SUMMARY.

An investigation has been made of the lunar inequalities from the Glasgow barograph records 1868–1912.

It is shown that an abnormal first and second harmonic exist in the inequalities obtained from barometrically quiet days, in addition to the non-periodic convexity effect. A provisional value of the normal lunar tide has been derived, which, for the total period of 45 years, has the value

$$0{\cdot}0110 \sin(2\theta + 60°) \text{ millibar,}$$

θ being reckoned from upper lunar transit.

REFERENCES TO LITERATURE.

BARTELS, J., 1927. "Über die atmosphärischen Gezeiten," *Veröff. Preuss. Met. Inst.*, Abh. 8, Nr. 9.

BECKER, L., 1925. "The Climatology of Glasgow," *M.O. Geophys. Mem.*, no. 23.

CHAPMAN, S., 1918. "The Lunar Atmospheric Tide at Greenwich, 1854–1917," *Quart. Journ. Roy. Met. Soc.*, vol. xliv, pp. 271–279.

CHAPMAN, S., 1936. "The Lunar Atmospheric Tide at Glasgow," *Proc. Roy. Soc. Edin.*, vol. lvi, pp. 1–5.

CHAPMAN, S., and AUSTIN, M., 1934. "The Lunar Atmospheric Tide at Buenos Ayres, 1891–1910," *Quart. Journ. Roy. Met. Soc.*, vol. lx, pp. 23–28.

ROBB, R. A., and TANNAHILL, T. R., 1935. "The Lunar Atmospheric Pressure Inequalities at Glasgow," *Proc. Roy. Soc. Edin.*, vol. lv, pp. 91–96.

(*Issued separately May* 1, 1939.)

IX.—Sources of Variation in Human Birth Weights. — By

H. P. Donald, Ph.D.(Edin.), Institute of Animal Genetics, University of Edinburgh. (With Three Graphs.)

(MS. received January 9, 1939. Revised MS. received February 22, 1939. Read March 6, 1939.)

AT times when processes of great ontogenetic importance are taking place, the interaction of heredity and environment presents problems of considerable interest. Laying as it does the foundations of future independent existence, prenatal development must be the concern of all those interested in the mental and physical suitability of human beings for the environment in which they have to live. Owing to lack of information, pregnancy from the point of view of the fœtus has perforce been largely neglected by the physician and sociologist, but the need to remedy this defect in our knowledge had been clearly indicated by Hogben (1931). As an example of a field of research that would repay much attention, the work of Pearson (1914) and S. Hansen (1920) may be mentioned. These authors wrote suggestively of the handicap of the first-born, using extensive data on the incidence of disease, mental defect, and the physical qualities of the new-born child in relation to birth order.

According to the Registrar-General (1936), the greatest single cause of infant mortality is "premature birth," a fact which assumes great importance in days of falling fertility, and which brings human experience into line with that of animals, the survival of which is closely bound up with the weight and strength of the newly born. Both from the individual and the population aspect, therefore, it is desirable to know more of the intimate association of mother and fœtus. From the nature of its data, the investigation now reported emphasizes the inequality of mothers rather than the varying response of fœtuses to the uterine environment, but it is to be recognized that birth weight may not only be controlled by the characteristics of uterine environment, but also be subject to genetical influences arising from within the fœtus itself.

The background of knowledge for the investigator of human birth weights consists of some few papers dealing with actual human birth weights and a considerable number devoted to the birth weights of animals. Of the former, only that of Toverud (1933) need be mentioned here. This paper is chiefly concerned to establish the importance of the seasonal food variation, and of the mothers' social status on the nutri-

tional condition of the new-born. Evidence is quoted and figures given to support the idea that the diet and occupation of the mother affect the weight of the new-born, but the question whether this is due entirely to modification of the length of pregnancy is not settled. This is a difficulty that besets all workers with human material, since any factor affecting the birth weight may do so by altering either the gestation period or the growth rate or both. In the present report, the main interest centres around the relative importance of such factors as season of birth, sex, and family. It will be shown that whereas the first two are of minor importance, the influence of family is so strong as to preclude the possibility that it might be accounted for by slight changes in the duration of pregnancy.

For the purposes of this study, it is desirable to mention certain papers representative of many dealing with two other types of investigation which have an indirect bearing on the problem here discussed. There is firstly the evidence of the waves of growth in the organs and tissues of man during gestation (Jackson, 1909; Bean, 1924; Schultz, 1926) and the critical stages through which the embryo passes (Stockard, 1920). The facts presented by these and other authors lead inevitably to the question of the relations between birth weight, stage of development (the cerebellum and intestines, for instance, are growing strongly during the last month of pregnancy), and subsequent development after the check to growth at birth. The second type of investigation deals with genetical differences in birth weight observed in animals. Species and breed crosses have been frequently made and the effects of heredity demonstrated. In some cases a large effect is claimed (Chapman and Lush, 1932, in sheep) and sometimes only a small one (Lush and others, 1934, in pigs). The classical genetical analysis of birth weights is that of Wright (1922) in guinea-pigs. A reorientation of the whole problem has been given by the work of Walton and Hammond (1938), who have clearly shown the importance of "maternal control" with the aid of a wide outcross between Shire horses and Shetland ponies. Whichever way the cross was made, the resulting foal was of a size appropriate to the dam's breed, and was not intermediate. There is, therefore, a division of hereditary effects to be made—(*a*) those arising from the genetic constitution of the fœtus, and (*b*) those arising from the genetic constitution of the mother and determining the physiological standards of her pregnancy.

Material and Methods.

Through the courtesy of the authorities concerned, access has been obtained to records of the Elsie Inglis Memorial Maternity Hospital, Edinburgh. It has been possible to utilize a mere fraction of the data

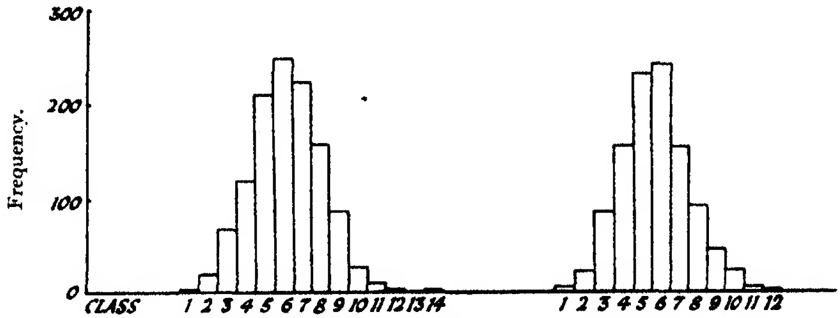
available, and there is a large fund of information waiting to be extracted from the detailed and well-kept records. Two main groups of birth weights have been collected: (*a*) individuals, (*b*) families. In the first, each weight has been classified according to the sex of the child, the month of birth, and the age of the mother at time of birth, for the last three years up to August 1938. Further sub-division was made to form groups of first-, second-, and later-born children. The second group (*b*) consists of cases where two or more records are available for the same family, and here also age of mother, sex, and date of birth of the children have been noted. Weights of the second group have also been included in the first and larger group, with the exception that no mother is represented twice in the later-born category, only the first available weight after the first birth being taken.

It is important to note that the data have been selected. Cases of maternal toxæmia, premature births when noted at time of delivery, and all still-births and abnormalities have been excluded. This is an important qualification since differing results obtained in separate investigations may be traced to the inclusion or otherwise of premature or post-mature births. Slight deviations from the usual term of pregnancy are difficult to detect, and consequently only cases where the term was considered to be more than a fortnight short have been excluded on account of prematurity.

Weight has been chosen in preference to length or the length-weight ratio as being more suitable for the measurement of variations in environment. Owing to the tendency for skeletal growth to proceed comparatively uniformly in spite of moderate environmental fluctuations, while the muscle and fat tissues respond more readily and act as a buffer protecting the vital parts from injurious fluctuations, weight should be a more sensitive indicator than length. On the same grounds, environmental effects which are detectable only with large numbers can hardly be important to the bulk of the children. Even in districts where poverty is marked, the size and growth of the children is not appreciably different from that in more prosperous areas (Paton and Findlay, 1926).

As an indication that the sample of records taken may be regarded as normally distributed about its mean, Graphs 1 and 2 have been drawn from the data on first-born males and females. Both distributions take the form of the normal curve, an observation which was made previously by Westergaard (1890). The females, however, showed a slight but significant positive skewness ($\sqrt{\beta_1} = 0{\cdot}311 \pm 0{\cdot}076$), which may mean that too many light-weight infants have been rejected as premature. These facts are important in the statistical treatment of the data and in

the interpretation of its results. The statistical methods used are those of Fisher (1936).



Males. Females.

GRAPH 1. GRAPH 2.

Class 1 4·5— 4·9 lb.
,, 6 7·0— 7·4 lb.
,, 14 11·0—11·4 lb.

GRAPHS 1 and 2.—Frequency distributions of weight at birth of first-born children.

SEASONAL CHANGES IN BIRTH WEIGHTS.

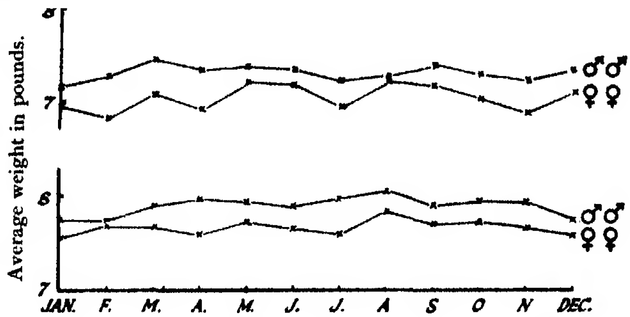
According to Toverud (1933), Norwegian mothers produced larger infants during late summer than at other times of the year, owing, she thinks, to the better supplies of fruit and vegetables. The extent to which this is true of Edinburgh can be judged from Table I, which shows the monthly averages for first- and second-born children, over a period of three years.

TABLE I.—SEASONAL CHANGES IN AVERAGE BIRTH WEIGHT (POUNDS).

Month.	First Born. Males.		First Born. Females.		Second Born. Males.		Second Born. Females.	
	No.	Average.	No.	Average.	No.	Average.	No.	Average.
January	98	7·17	89	6·96	35	7·33	34	7·41
February	85	7·30	74	6·82	31	7·57	40	7·17
March	88	7·47	93	7·09	33	7·53	40	7·16
April	115	7·37	95	6·95	30	7·78	23	7·75
May	132	7·40	94	7·21	42	7·36	25	7·39
June	90	7·36	99	7·22	26	7·39	24	7·60
July	96	7·26	95	6·99	31	7·97	35	7·70
August	88	7·30	93	7·23	43	7·44	29	7·57
September	91	7·41	85	7·20	30	7·67	29	7·55
October	96	7·33	81	7·06	30	7·93	35	7·35
November	84	7·27	77	6·92	33	7·70	26	7·79
December	98	7·37	78	7·14	32	7·66	37	7·62
Totals	1161	7·336	1053	7·073	396	7·60	377	7·48

Graph 3 shows these results in a more easily appreciated form. It suggests that among first-born children males are regularly heavier than females, and less affected by seasonal fluctuations in average weight.

The second-born children, owing to the fewer numbers, showed greater variability of average weight from month to month, and no clear difference in weight according to sex. These anticipations are borne out by



Above—Edinburgh data, first-born only.
Below—Oslo data (Toverud, 1933), all births.

GRAPH 3.—Seasonal distribution of average birth weight (in pounds) for Edinburgh and Oslo (Toverud).

the analysis of the monthly means in Table II. The three years' observations are treated separately for the first-born, but combined for the second-born children.

TABLE II.—ANALYSIS OF THE VARIANCE OF MEAN BIRTH WEIGHTS ACCORDING TO SEX, MONTH AND YEAR.

A. *First-born Children.*

Source of Variation.	D.F.	Sum of Squares.	Mean Square.	F.* Observed.	F.* 5 per cent. Point.	F.* 1 per cent. Point.
Sex . . .	1	1·254	1·254	47·7	4·0	7·1
Year . .	2	0·035	0·018	..	..	..
Month . .	11	0·611	0·056	2·1	2·0	2·6
Remainder .	57	1·497	0·026	..	..	..
Total . .	71	3·397	..	..	..	..

B. *Second-born Children.*

Source of Variation.	D.F.	Sum of Squares.	Mean Square.	F.* Observed.	F.* 5 per cent. Point.	F.* 1 per cent. Point.
Sex . . .	1	0·069	0·069	2·3	4·8	..
Month . .	11	0·631	0·057	1·9	2·8	..
Remainder .	11	0·331	0·030	..	..	..
Total . .	23	1·031	..	..	..	..

* According to Snedecor (1934).

On these figures, sex and month of birth have an established effect on the weight of first-born children, but not year of birth. Neither sex nor month have a detectable effect on the means for the less numerous second-born.

Graph 3 shows that the influence of month of birth affects chiefly first-born females during the summer months May to September. The

anomalous fall in the average during July is due to an unexplained low average in one of the three years. Toverud's monthly averages show very little change except for a rise in August and a fall in December and January. Combining the evidence of both sets of data, it may be supposed that winter babies are sometimes slightly lighter and summer babies sometimes slightly heavier than the yearly average.

Effect of Age of Mother.

In the literature there is a certain amount of disagreement concerning the influence of age of mother on weight of offspring. Duncan (1871) and Toverud (1933) reported an association between the age of the mother and the weight of the child. H. J. Hansen (1913) and Heiberg (1911) hold the view that birth order and not age of mother accounts for the increasing weight of later babies. Neither factor is, of course, very useful as an explanation of why later babies are heavier. The present results are shown in Table III. They are confined to first and second births only, for which adequate numbers are available.

Table III.—Age of Mother, and Birth Weight.
(Number of births in brackets.)

Age of Mother.	First Born.		Second Born.		Sex Ratio. First to Fifth Births. ♂♂ per 100 ♀♀.
	Males.	Females.	Males.	Females.	
15–19 *	7·27 (95)	6·95 (89)	7·55 (4)	7·20 (5)	106
20–24	7·35 (483)	7·05 (386)	7·65 (109)	7·39 (99)	122
25–29	7·31 (337)	7·11 (321)	7·65 (166)	7·54 (141)	108
30–34	7·34 (114)	6·94 (120)	7·47 (86)	7·60 (94)	95
35–42	7·56 (23)	7·24 (33)	7·44 (31)	7·22 (38)	72
All mothers	7·33 (1052)	7·05 (949)	7·60 (396)	7·48 (377)	108

* Including two mothers of 13 and 14 whose first female and male infants weighed each 6·5 lb.

So far as can be judged from the Edinburgh data, the weight of first-born children of either sex is not appreciably affected by the age of the mother. The weights of second children, on the other hand, show that both males and females increase in weight to a mother's age of about 25–30 years and then decrease again. This is in general agreement with the observations of Pearson, Duncan, and H. J. Hansen.

As a matter of interest, the sex ratio of children of mothers of different ages has been attached to Table II. The figures are hardly large enough to be of much value, but the occurrence of a low proportion of males at either end of the reproductive life of women has also been noticed by H. J. Hansen (1913). If it is true that, under good environmental conditions, the sex ratio rises, the low sex ratios must mean that the fœtuses of

the young and old mothers were exposed to more hazardous conditions than those of mothers of intermediate ages.

Influence of Time elapsing between Births on Birth Weight.

No reference to a previous examination of this possible influence has been seen. Yet if the growth of a fœtus depends on the supply of nutriment from the mother, it might be expected that the occurrence of a pregnancy immediately after a parturition and lactation would find the reserves of the mother at a low ebb to the detriment of the fœtus. There is some evidence bearing on this point from experiments with sheep, in which it was observed that breeding from immature animals resulted in retarded growth and development of both ewe and lamb (Griswold, 1930). It may be different in rodents. Cole and Hart (1938) found that in rats pregnancy stimulated growth, so that pregnant females were heavier than non-bred control females. Cattle, unlike other domestic animals, are expected to support a growing fœtus as well as to milk heavily during all but two months or less of a nine-month pregnancy.

For analysis of the differences in weight and time between pairs of children adjacent in the birth order, pairs of males and pairs of females have been chosen. Owing to the tendency usual in this type of data for values on either side of the mean to be associated with paired values which are closer to the mean (regression), differences between birth weights when the first is low are likely to be positive, whereas those between birth weights when the first is high are likely to be negative. In order to prevent this tendency obscuring the effect (if any) of the months between births, the first of the two births have been classified into five weight classes, a procedure which enables the effect of time between births to be estimated within each weight class for first births.

Table IV.—Time in Months (Time) and Difference in Weight (Diff.) between First and Second Births.

Weight of First Child.	Up to 5·9 lb.		6·0–6·9 lb.		7·0–7·9 lb.		8·0–8·9 lb.		9·0 lb. and over.		Total.	
	Time.	Diff.	Time.	Diff.	Time.	Diff.	Time.	Diff.	Time.	Diff.	Time.	Diff.
Means—(1) Both males	35·1	+1·45	33·2	+0·48	32·9	+0·35	34·8	−0·32	36·6	−0·83	33·8	+0·21
(2) Both females	28·1	+1·07	31·7	+0·29	39·9	+0·17	33·2	−0·11	26·0	−0·43	34·7	+0·23
Number of families—												
(1) Both males	8		34		57		36		8		143	
(2) Both females	7		34		38		9		4		92	

Table IV provides a general picture of the situation and shows how the second births tend to be heavier than light first births, but lighter than heavy first births, the tendency over all births together, however, being slightly positive. The question whether time between births affects the weight of the second of a pair can now be answered from the analysis given in Table V, which follows the method of Fisher (1936).

TABLE V.—ANALYSIS OF VARIANCE AND COVARIANCE OF TIME IN MONTHS (TIME) AND DIFFERENCE IN WEIGHT (DIFF.) IN POUNDS BETWEEN SUCCESSIVE BIRTHS.

Source of Variation.	D.F.	(Time)².	(Time × Diff.).	(Diff.)².	D.F.	Regression Sum Squares.	Adjusted (Diff.)².	Mean Square.	F.* Observed.	F.* 1 per cent. Point.
A. *Both Males.*										
Total .	142	44244	+ 88·1	113·1	1 141	0·18	112·9	0·80		
Within Wt. Classes.	138	44076	+ 129·4	78·5	1 137	0·38	78·1	0·57		
Between Wt. Classes.	4	168	− 41·3	34·6	1 3	10·15	24·4	8·14	14·28	3·92
B. *Both Females.*										
Total .	91	35618	+ 139·2	63·4	1 90	0·54	62·9	0·70		
Within Wt. Classes.	87	33661	+ 169·6	55·4	1 86	0·85	54·5	0·63		
Between Wt. Classes.	4	1957	− 30·4	8·0	1 3	0·47	7·5	2·50	3·97	4·02

* See Snedecor (1934).

From this table it may be concluded that (1) the mean differences between births vary significantly according to the weight class of the first birth; in other words, there is a significant tendency for the second of a pair of births to be closer than the first to the mean of all births whenever the first has departed from that mean; (2) that an effect of months between births is not established by these data, since the regression sum of squares is not distinctly larger than the corresponding mean square; (3) that there is no relation between the time between births and the weight of the first of a pair. This conclusion can be derived from the first two columns of Table V. A further comparison of the differences in weight between first and third or fourth child in relation to the time elapsing between their births has led to similar results.

Effect of Order of Birth on Birth Weights.

The extent of this effect has considerable importance from the fact that a relatively low birth weight among first-born children may have some connection with the mental and physical handicap of the first-born (Pearson, 1914; Hansen, S., 1920). Most investigators agree that there is an increase in weight up to the third child. What happens after that has not been satisfactorily determined owing to the inadequacy of the available numbers, but it seems likely that there is but slight change from the fourth onwards. A comparison of the results now presented with some previously obtained is afforded in Table VI. References to further data are given by Heiberg (1911).

Table VI.—Effect of Order of Birth according to Various Investigations.

Order of Birth.	1.	2.	3.	4.	5.	6.	7.	8.	9–10.	Total Number of Cases.
(1) Ingerslev, Denmark, ♂♂	7·28	7·56	7·68	7·71	7·86					1833
♀♀	7·07	7·41	7·30	7·43	7·59					1617
(2) Hansen, Denmark, ♂♂	7·74	8·10	8·34	8·32	8·49	8·47	8·45	8·43	8·49	3005
♀♀	7·49	7·80	7·94	8·14	7·95	7·98	8·16	8·07	8·05	2818
(3) Toverud, Norway, ♂♂	7·69	←		→	8·16	←			→	2205
♀♀	7·52	←		→	7·88	←			→	2046
(4) Duncan, Scotland, ♂♂ ♀♀	7·20	7·31	7·35	7·19	7·45	7·32	7·31 (7 and over)			2087
(5) Pearson, England, ♂♂	7·01	7·36	7·41		7·70		7·91		7·59	856
♀♀	6·76	7·08	7·33		7·36		7·32		7·65	866
(6) Donald, Scotland, ♂♂	7·33	7·60	7·78	7·87						1571
♀♀	7·05	7·48	7·47	7·42						1465

All the data, with the exception of Hansen's, come from city populations. Hansen's were not collected at clinics, but by midwives in country districts. It is therefore difficult to judge whether the variations in average weight of the six series are attributable to the populations from which they were derived or to the methods of obtaining the data. The Norwegian and provincial Danish averages are, however, suggestive of real differences from the remaining city data. All the series agree in showing an increase in both sexes up to the third child. The males may thereafter show further slight increases, but it is not advisable to draw conclusions, because the numbers are small and because of the increasing proportion of births contributed by the poorer classes and parents of possibly sub-average physique.

It will be observed that all the data given in Table VI agree that the second-born female is slightly larger in general than the first-born male.

This permits an estimate of the importance of birth order as a source of variation in birth weight as compared with sex, namely, that the difference between first- and second-born children is equal to or slightly greater than that between males and females, but that after the second birth, birth order has a much smaller effect than sex.

Another point of interest (which has also been studied by Goldfeld, 1912) is whether the weight of females following males is different from that of females following females. The sources quoted in Table VI do not provide this information, so that only the present Edinburgh data can be given.

TABLE VII.—MEAN WEIGHTS OF FIRST- AND SECOND-BORN INFANTS (IN POUNDS).

	First.	Second.	Difference.	Number.
(1) Both males . .	7·471	7·655	+0·184 ± 0·113	143 pairs.
(2) Both females .	7·072	7·338	+0·266 ± 0·130	92 ,,
(3) Male-female .	7·461	7·539	+0·078 ± 0·127	119 ,,
(4) Female-male .	7·146	7·645	+0·499 ± 0·115	100 ,,
(5) All males . .	7·466	7·651	+0·185 ± 0·081	262; 243 ,,
(6) All females . .	7·110	7·450	+0·340 ± 0·089	192; 211 ,,

Females following males are larger than females following females according to these means, but the difference ($0·198 \pm 0·135$) is not significant. Second-born males of both classes have practically the same mean weight. The change in uterine conditions responsible for the increased weight of second and later births is therefore largely independent of the sex and weight of the first fœtus.

SIMILARITY OF BIRTH WEIGHTS WITHIN FAMILIES.

Although obstetricians now commonly consider the birth weights as significant features of family histories, there appears to be little published evidence concerning the extent of the resemblance in birth weight of children of the same family. Apart from the obstetrical importance of such information, it is desirable for sociological reasons to know a great deal more of the relation between birth weight and subsequent mental and physical development, and of the extent to which birth weight is an individual and familial characteristic. For this latter problem, the family histories made available may serve as a preliminary source of information.

Since most of the families consist of only the first two children, they will be considered first. Such families can be naturally divided into four groups—male-male, male-female, female-male, and female-female. It has been thought well to deal with them separately, since both sex and order of birth have an established effect, at least on the first two births. Within these groups there was naturally much variation in birth weight,

which is measured by the total variance of Table VIII. This total variance has been subdivided into three portions representing the contributions due to (1) birth order and sex, (2) differences between the family averages, and (3) the remaining unknown causes of variation. Thus in the 454 families concerned, about 75 per cent. of the variation

TABLE VIII.—ANALYSIS OF VARIANCE IN FAMILIES OF TWO (FIRST TWO ONLY).

	Male-male.	Male-female.	Female-male.	Female-female.
Total variance	242·21	227·09	143·85	144·31
Variance between births (B)	2·42	0·27	12·45	3·27
Degrees of freedom	1	1	1	1
Mean square	2·42	0·27	12·45	3·27
Per cent. of total variance	1·00	0·12	8·65	2·27
Total variance less (B)	239·79	226·82	131·40	141·04
Degrees of freedom	284	236	198	182
Mean square	0·84	0·96	0·66	0·77
Variance between families	181·34	171·38	102·70	110·19
Degrees of freedom	142	118	99	91
Mean square	1·28	1·45	1·04	1·21
Per cent. of total variance	74·87	75·47	71·39	76·36
Remainder	58·45	55·44	28·70	30·85
Degrees of freedom	142	118	99	91
Mean square	0·41	0·47	0·29	0·34
Per cent. of total variance	24·13	24·41	19·95	21·38
Intra-class correlation	+0·51	+0·51	+0·56	+0·56

in birth weight arose from differences among the family averages instead of 50 per cent., which would be expected in families of two if there were no tendency for sibs to be more alike than non-sibs. Compared with this, the amount due to sex and order of birth (in which most previous investigators have been interested) is almost negligible. Where the children were of the same sex, birth order was responsible for only 1 per cent. (males) and 2·27 per cent. (females) of the total variance. The interaction of birth order and sex in the other two groups accounts for the lower and higher values (0·12 per cent. and 8·65 per cent.). The statistical significance of each controlled source of variation may be judged by comparing the mean squares derived for each with the error mean square obtained from the remainder variance within each of the four groups. In the families of two males, the mean square for differences due to birth order is 2·42 with one degree of freedom, which is to be compared with the error mean square of 0·41 with 142 degrees of freedom. According to Snedecor's method (1934), the ratio $2{\cdot}42/0{\cdot}41 = 5{\cdot}9$ is compared with the tabled value of 3·9 for the same degrees of freedom. The tabled value gives the lower limit of all values which would occur by chance only once in twenty samples, and since the observed value is

greater, the chance that the observed effect of birth order is merely an accident of sampling is less than 5 per cent. By the same process it may be shown that the effect of birth order in the families of two females and the combined effect of birth order and sex in the female-male families are also significant. In the male-female families the effects of sex and birth order cancel each other, so that the net effect on the variance is non-significant. Similarly, in each group, the mean squares between families give, with the corresponding error mean squares, a ratio of about 3, a value which exceeds the 1 per cent. point for the appropriate degrees of freedom, so that the effects of family may be considered very significant.

Similar calculations have been made for families of three and four children. From the discussion of birth order above, it will be expected that as the family becomes larger the significance of birth order will diminish, and indeed it can only be demonstrated in the large group of families of three of mixed sex, in which it cannot be easily separated from the influence of sex. In respect of family, however, what has been found true for families of two holds as strongly for families of three and four (Table IX).

TABLE IX.—THE SIGNIFICANCE OF BIRTH ORDER AND FAMILY IN FAMILIES OF THREE AND FOUR.

	Number.	Mean Square.			Significance.	
		Birth Order.	Family.	Error.	Birth Order.	Family.
Families of three—						
All males .	21	1·29	1·71	0·52	NS	SS
All females .	13	0·19	1·01	0·34	NS	S
Mixed sex .	82	1·80	2·11	0·42	S	SS
Families of four—						
Mixed sex .	35	0·14	2·61	0·54	NS	SS

NS, non-significant; S, significant at 5 per cent. point; SS, significant at 1 per cent. point.

The resemblance between sibs accounting for the significance of family may be expressed in terms of an intra-class correlation (Fisher, 1936) calculated for Table VIII, by comparing the reduction in the value of the mean square due to eliminating the effects of family with the original value. In the male-male families this is $(0{\cdot}84 - 0{\cdot}41)/0{\cdot}84 = +0{\cdot}51$. The values obtained show that the environmental and genetical factors concerned occur in combinations which are as strongly characteristic of families as those determining, say, mature height. Comparison of the mean squares within families ("remainder" in Table VIII) and between families shows that the latter (and therefore the estimates of sib correlation) are strongly significant. This conclusion has been checked by

making similar calculations for other groups of children, namely, first and third, second and third, first and fourth, families of three and families of four. The correlations are given in Table X. Allowing for the variation in the coefficients owing to the smaller numbers of families, it will be seen that the estimate of sib resemblance is approximately 0·5. A combined estimate based on the independent combinations of Table X yields the value 0·48.

TABLE X.—INTRA-CLASS CORRELATIONS MEASURING FAMILY RESEMBLANCE IN BIRTH WEIGHT. VARIANCE DUE TO BIRTH ORDER AND SEX REMOVED UNLESS OTHERWISE STATED. NUMBERS OF PAIRS OR FAMILIES IN BRACKETS.

Independent Combinations.	Correlation Coefficient.	Derived Combinations.	Correlation Coefficient.
Families of two—		Two children—	
Male-male	0·51 (143)	(1) First and third:	
Male-female	0·51 (119)	Male-male	0·57 (35)
Female-male	0·56 (100)	Male-female	0·51 (35)
Female-female	0·56 (92)	Female-male	0·39 (33)
		Female-female	0·26 (32)
Families of three—			
Mixed sex	0·55 (82)	(2) Second and third:	
All males	0·44 (21)	Male-male	0·68 (52)
All females	0·40 (13)	Male-female	0·37 (45)
		Female-male	0·79 (48)
		Female-female	0·63 (46)
Familes of four—			
Mixed sex	0·49 (35)	(3) First and fourth, Second and fifth:	
		Sex disregarded	0·55 (35)

DISCUSSION.

The environmental and genetical characteristics of each family cannot, unfortunately, be completely separated. The only conjectures that may be made about their relative importance have to be based on the observed effects of those variations in environment which can be measured. These may be classed as external to both mother and child, for example, season of birth; and as internal to mother but external to child, for example, order of birth and age of the mother. None of these has an influence comparable with that of family. The only genetical difference which can be measured is that of sex, and here also the effect is relatively small. Bearing in mind the known ability of the fœtus in various mammalia to grow at the expense of the mother's own body weight, if necessary, it may be supposed that the mother may smoothe out the favourable and unfavourable impacts of environment. There remain the largely unaccounted for family differences which must, in the meantime, be attributed to the genetical constitution of the children and to the genetical constitution of their mother. The former determines the growth reactions

of the fœtuses to their uterine environment, whereas the latter determines the quality of that environment. The extent to which children are of similar weight at birth because they have grown in the same uterus cannot, as yet, be directly compared with the extent of their genetic similarity. Experiments with laboratory and domestic animals have established the influence of both "maternal control" (Walton and Hammond, 1938) and the genetical constitution of the offspring (Wright, 1922) in determining birth weight. Their relative importance varies according to circumstances, and the position in man is difficult to anticipate.

One characteristic of birth weights in general, the almost normal distribution of the weights, has a certain bearing on the matter. It may be supposed that if fœtal increases in size were subject to a law of diminishing response to increments of food supply, the distribution of weights at birth would show a skewness on account of a shortage of heavy births. This does not appear to be the case. The number of large babies is as great as would be expected on a purely random assortment of favourable genetic and environmental factors with additive interaction, and consequently it may be inferred that the capacity of the fœtus to grow is, in general, not limited by its own diminishing response to favourable changes in the environment. Unless there is a tendency for both kinds of factors to act in the same direction (which may well happen since mating is selective), average birth weight must fall short of the maximum which it might reach. Thus a maternally controlled uterine environment could be the limiting factor in respect of average birth weight.

On the other hand it is to be expected that a sib correlation of 0·5 would be found in a character exhibiting purely genetic variation. With random mating, half the genetic variation would occur within families, but the fact that the observed variation within families approaches this figure does not necessarily mean that such a character is involved. Furthermore, the interpretation of such estimates of family resemblance is rendered more difficult by the absence of information about the effects of sampling. When a fairly uniform social group is used as the basis of the calculations, as is the case here, the applicability of the results to the population as a whole depends on whether the intra-familial differences in other social groups are of the same order. For a discussion on this subject, Hogben (1933) should be consulted.

Evidence from identical and non-identical twins in respect of birth weight has to be regarded with caution, since the assumption that the fœtal nutrition of both classes is subject to essentially the same conditions may not be tenable. In the meantime, however, it may be considered significant that Essen-Möller (1930) finds no greater resemblance in birth

weight among one-egg twins than among two-egg twins, although there is a greater resemblance in length. This bears out the supposition that skeletal growth is less affected by environment than the fat and muscle tissues, the plasticity of which protects bone growth, and that the genetical constitution of the fœtuses has a lesser influence on the birth weight than the genetical constitution of the mother in so far as the latter affects fœtal nutrition.

It may be useful to withdraw, in conclusion, some distance from the immediate problem of variation in birth weight, and regard it in relation to more general problems. Not only for obstetrical reasons, but for considerations of mortality, fertility, and post-natal physical development, birth weight has an inherent value. It is not a direct and unequivocal measure of growth, but at present what is lost in meaning is counterbalanced by the accessibility and scope of the material. To a considerable degree, birth weight represents the nine-months growth of tissues and organs in a particular type of environment. Of the growth of man in general it is known that the rate of development of the various parts relative to the whole and the remaining parts undergoes very great changes with the passage of time (Schultz, 1926). Imposed on these changes are the three phases of growth with their maxima and minima (Bean, 1924). When a child is born, that is, when it makes the change-over suddenly to a new mode of nutrition and respiration, the event must therefore be considered in relation to the stage of development which it has reached, for it is clear that shock and check might have far-reaching consequences when birth interrupts a stage of rapid growth such as a fœtus of five pounds weight would be commencing. As there seems to be no reason at present to suppose that there is any sharp distinction between "normal" full-term infants and premature or immature infants, it seems reasonable to assume that some of the dangers of child-birth to the child gradually diminish as the birth weight increases, and that a part at least of the disadvantage of the premature child is shared by the full-term child in some degree determined by its weight. This consideration does not, of course, apply to those difficulties of parturition which are increased by weight and size, but to the capacity of the child to adjust itself to the new conditions. Thus if, as Capper (1928), Sunde (1930), and Brander (1938) maintain on the basis of their extensive investigations, obviously premature and immature infants are subject to a heavy mortality, are on the average subnormal mentally and physically, and are more prone to nervous and respiratory disorders, it would not be surprising if these characteristics became less marked as the average birth weight for the population as a whole were approached and exceeded. For the same reason it is prob-

ably worth while investigating the physical and mental development of those families in which low birth weights occur regularly, since they may conceivably contribute more than their share of those unfortunates who are obliged to accept the hospitality of the State.

SUMMARY.

1. An examination has been made of the factors affecting the birth weight of rather more than 3000 infants born at term of healthy mothers. The frequency distributions of the weights of first-born males and females showed a close approximation to normal curves (Graphs 1 and 2).

2. Monthly average weights show small but significant differences, tending to be higher in summer than in winter (Table II, Graph 3).

3. The effect of sex on birth weight is greater than that of month of birth, is about the same as that of birth order in small families, but accounts for comparatively little of the total variation (Tables II and VIII).

4. Age of mother had no apparent effect on the weight of first-born children, but may have had a slight influence on second-born children, mothers about thirty years old having the heaviest (Table III).

5. The time elapsing between births had no demonstrable influence on birth weight (Tables IV and V).

6. Data from six investigations, including the one now reported, agree that both sexes show an increase in weight up to the third child. The averages for later children are not consistent (Table VI).

7. Analysis of the observed total variation in birth weight indicates that whereas sex and birth order acting in the same direction cause less than 10 per cent., and much less when acting in opposite directions, differences between families of four or fewer children account for at least half the variation (Table VIII).

8. It is not possible with these data to determine directly how far birth weight as a family characteristic is dependent on the similarity of genetic constitution of the sibs as opposed to the constancy of their prenatal environment, but from various considerations the view is favoured that birth weight is a quality of the mother rather than of the children.

9. The significance of birth weight in relation to subsequent mental and physical development is briefly discussed.

ACKNOWLEDGMENT.

The author wishes to acknowledge with thanks the courtesy and assistance of the authorities of the Elsie Inglis Memorial Maternity Hospital, Edinburgh, Dr Susanne Paterson in particular, during the collection of the data.

REFERENCES TO LITERATURE.

BEAN, R. B., 1924. "The Pulse of Growth in Man," *Anat. Rec.*, vol. xxviii, pp. 45–61.

BRANDER, T., 1938. "Kann die Konstitution durch Frühgeburt verändert werden?" *Zeits. mensch. Vererb. u. Konstitutionl.*, vol. xxii, pp. 61–68. (Quoted from *Resumptio Gen.*, vol. xiii, p. 65, 1938.)

CAPPER, A., 1928. "The Fate and Development of the Immature and of the Premature Child," *Amer. Journ. Dis. Children*, vol. xxxv, pp. 262–288, 433–491.

CHAPMAN, A. B., and LUSH, J. L., 1932. "Twinning, Sex-ratio, and Genetic Variability in Birth Weight in Sheep," *Journ. Hered.*, vol. xxiii, pp. 473–478.

COLE, H. H., and HART, G. H., 1938. "The Effect of Pregnancy and Lactation on Growth in the Rat," *Amer. Journ. Physiol.*, vol. cxxiii, pp. 589–597.

DUNCAN, J. M., 1871. *Fecundity, Fertility, and Sterility*, A. & C. Black, Edinburgh.

ESSEN-MÖLLER, E., 1930. "Ähneln sich zweieiige Zwillinge mehr als eineiige Zwillinge in Bezug auf Gewicht und Länge bei der Geburt?" *Acta obstet. gynecol. scand.*, vol. ix, pp. 107–123.

FISHER, R. A., 1936. *Statistical Methods for Research Workers*, 6th ed., Oliver & Boyd, Edinburgh.

GOLDFELD, Z., 1912. "Die Abhängigkeit der körperlichen Entwicklung Neugeborener vom Beruf ihrer Eltern, *Zeits. Geburtshilfe*, vol. lxxii, p. 407.

GRISWOLD, D. J., 1930. "A Study of Breeding Ewe Lambs," *Proc. Amer. Soc. An. Prod.*, pp. 106–108.

HANSEN, H. J., 1920–28. (First issued 1913.) "Undersøgelser over nyfødte Børns Vaegt," *Medd. Danmarks Antropologi*, vol. ii, pp. 1–110.

HANSEN, S., 1920–28. "Om de først fødte Børns ringere Kvalitet," *Medd. Danmarks Antropologi*, vol. ii, pp. 111–152.

HEIBERG, P., 1911. "Om Vejninger af nyfødte Børn," *Medd. Danmarks Antropologi*, vol. i, pp. 337–350.

HOGBEN, L., 1931. *Genetic Principles in Medicine and Social Science*, Allen & Unwin, London, 230 pp.

HOGBEN, L., 1933. *Nature and Nurture*, Norton, New York, 143 pp.

JACKSON, C. M., 1909. "On the Pre-natal Growth of the Human Body and the Relative Growth of the Various Organs and Parts," *Amer. Journ. Anat.*, vol. ix, pp. 119–166.

LUSH, J. L., HETZER, H. O., and CULBERTSON, C. C., 1934. "Factors Affecting Birth Weights of Swine," *Genetics*, vol. xix, pp. 329–343.

PATON, N., and FINDLAY, L., 1926. "Poverty Nutrition and Growth," *Med. Res. Council Rept.*, H.M. Stationery Office.

PEARSON, K., 1914. "On the Handicapping of the First-Born," *Galton Lab. Publ.*, Dulau, London.

REGISTRAR-GENERAL, 1936. *The Registrar-General's Statistical Review of England and Wales for the Year 1936*, Part I, H.M. Stationery Office.

SCHULTZ, A. H., 1926. "Fœtal Growth of Man and Other Primates," *Quart. Rev. Biol.*, vol. i, pp. 463–521.

SNEDECOR, G. W., 1934. *Calculation and Interpretation of Analysis of Variance and Covariance*, Collegiate Press, Ames, Iowa.

STOCKARD, C. R., 1920. "Developmental Rate and Structural Expression: An Experimental Study of Twins, 'Double Monsters,' and Single Deformities, and the Interaction among Embryonic Organs during their Origin and Development," *Amer. Journ. Anat.*, vol. xxviii, pp. 115–277.

SUNDE, A., 1930. "Die Prognose der Frühgeborenen und die Prophylaxe des Geburtstraumas," *Acta. obstet. gynecol. scand.*, vol. ix, pp. 477–553.

TOVERUD, K. U., 1933. "Nutritional Condition of New-Born Infants," *Amer. Journ. Dis. Children*, vol. xlvi, pp. 954–968.

WALTON, A., and HAMMOND, J., 1938. "The Maternal Effects on Growth and Conformation in Shire Horse-Shetland Pony Crosses," *Proc. Roy. Soc. Lond.*, B, vol. cxxv, pp. 311–335.

WESTERGAARD, H., 1890. "Statistikens Theori." (Quoted by Heiberg.)

WRIGHT, S., 1922. "The Effects of Inbreeding and Crossbreeding on Guinea-Pigs. I. Decline in Vigour. II. Differentiation among Inbred Families," *U.S. Dept. Agric. Bull.* 1090, pp. 1–63. III. "Crosses between Highly Inbred Families," *U.S. Dept. Agric. Bull.* 1121, pp. 1–60.

(*Issued separately May* 1, 1939.)

X.—On the Invariance of Quantized Field Equations. By

K. Fuchs, Ph.D., University of Edinburgh. *Communicated by* Professor MAX BORN, Hon. D.Sc.

(MS. received December 5, 1938. Read January 9, 1939.)

1. INTRODUCTION.

HEISENBERG and Pauli (1929) developed a general scheme for the quantization of a field, if the field equations can be derived from a variation principle,

$$\delta\int L(z^a, z_i^a)dx^1dx^2dx^3dx^4=0. \qquad (1)$$

Here z^a denotes the field variables, L is the Lagrangian, and

$$z_i^a=\frac{dz^a}{dx^i}. \qquad (2)$$

The scheme of Heisenberg and Pauli is known to be Lorentz invariant. It is the purpose of this paper to show that it is also invariant with regard to all co-ordinate transformations allowed by the general theory of relativity. The method adopted to prove this is that used by Infeld (1937) and Pryce (1937) to prove the invariance of the "New Field Equations" against Lorentz transformations.

In order to formulate the quantum mechanical commutation brackets, it is necessary to define the canonically conjugate field variables. In the Lorentz invariant theory of Heisenberg and Pauli this can easily be done by the definition

$$P_a=\frac{\partial L}{\partial z_4^a}. \qquad (3)$$

Here $x^4=ct$ is the time co-ordinate. If we use a general metric, the time co-ordinate can in general be separated only locally. There are now two alternatives. We may demand that the canonically conjugate variable has invariant significance. In that case we could proceed by defining the canonically conjugate variables at any given point in a local Lorentz system at that point. It is obvious that such a procedure would involve very cumbersome calculations, since we would have to consider

a co-ordinate system which varies from point to point. The alternative is to assume that the canonically conjugate variable has no invariant significance. We define similarly to (3) the tensor density

$$P^i_\alpha = \frac{\partial L}{\partial z^\alpha_i}. \qquad (3a)$$

Then the canonically conjugate variables do not form a tensor density, but they are only part of the components of P^i_α, namely, the components P^4_α. In this case the field equations cannot be written as equations between tensors or tensor densities. For, if we change from one co-ordinate system to another, the canonically conjugate variables will in general be different components of the same tensor density P^i_α. The field equations will, therefore, be only components of tensor or tensor-density equations. It is, of course, to be expected that the space- or time-like character of these equations should be preserved, so that in the limiting case the equations reduce to those of Heisenberg and Pauli. With this restriction, the principle of general invariance demands that the various components of the field equations are equivalent.

A point of view intermediate between the two alternatives has recently been adopted by Weiss (1938). (My thanks are due to Dr P. Weiss for the privilege of seeing his paper before publication.) He assumes a plane to be given, and the normal to this plane gives a direction independent of the co-ordinate system. Taking the component of P^i_α (with respect to the index i) along this direction, the canonically conjugate variables can be defined without reference to any particular co-ordinate system. The usual commutation relations are then adopted in this plane. However, this is not a complete proof of the invariance of quantum dynamics; if we adopt the commutation relations in one plane only, it is necessary to show that—as a consequence of the field equations—they also hold in any other plane. If we adopt the commutation relations in any arbitrary plane, the problem is overdetermined, and it must be shown that the system of commutation relations plus field equations is not contradictory.

In this paper I shall adopt the second alternative.

2. Canonical Form of the Field Equations.

In the general theory the Lagrangian function L must be a scalar density,

$$L = \sqrt{(-g)}\mathbf{L}, \qquad (4)$$

where $\mathbf{L}$ is a scalar function and g the metric determinant. The field

variables z^a may be components of a tensor of any rank. If we transform from one co-ordinate system to another,

$$\bar{x}^i = \bar{x}^i(x^k), \qquad (5)$$

the field variables transform according to an equation

$$\bar{z}^a = \mathbf{A}^a_\beta z^\beta, \qquad (6)$$

where the coefficients $\mathbf{A}^a_\beta$ can be calculated from the coefficients $d\bar{x}^i/dx^k$ of the transformation, if the rank of the tensor z^a is known. The derivatives of the z^a, of course, do not form a tensor. They transform according to the law

$$\bar{z}^a_i = \mathbf{A}^a_\beta \frac{dx^k}{d\bar{x}^i} z^\beta_k + \frac{d\mathbf{A}^a_\beta}{dx^k} \frac{dx^k}{d\bar{x}^i} z^\beta. \qquad (6a)$$

We now define

$$\mathrm{P}^i_a = \frac{\partial \mathrm{L}}{\partial z^a_i}. \qquad (7)$$

The P^i_a form the components of a tensor density, in spite of the fact that the z^a_i are tensors only for affine transformations. This is the reason why it is possible to investigate the invariance of quantum dynamics not only against Lorentz transformations, but also against the group of transformations allowed by the general theory of relativity, without explicitly introducing the coefficients of affine connection.

The P^i_a in the new co-ordinate system are

$$\bar{\mathrm{P}}^i_a = \frac{\partial \bar{\mathrm{L}}}{\partial \bar{z}^a_i} = \frac{\sqrt{(-\bar{g})}}{\sqrt{(-g)}} \frac{\partial \mathrm{L}}{\partial z^\beta_k} \frac{\partial z^\beta_k}{\partial \bar{z}^a_i}.$$

For, according to (4), we have $\bar{\mathrm{L}} = \mathrm{L}\sqrt{(-\bar{g})}/\sqrt{(-g)}$. If we define the coefficients $\mathbf{B}^a_\beta$ by the equation

$$\mathbf{A}^a_\gamma \mathbf{B}^\gamma_\beta = \delta^a_\beta,$$

then it follows from (6*a*) that

$$\frac{\partial z^\beta_k}{\partial \bar{z}^a_i} = \mathbf{B}^\beta_a \frac{d\bar{x}^i}{dx^k},$$

and the transformed P^i_a are

$$\bar{\mathrm{P}}^i_a = \frac{\sqrt{(-\bar{g})}}{\sqrt{(-g)}} \mathbf{B}^\beta_a \frac{d\bar{x}^i}{dx^k} \mathrm{P}^k_\beta. \qquad (8)$$

This equation shows that indeed the P^i_a transform like a tensor density.

The field equations follow from the variation principle (1) in the form of the Euler equations,

$$\frac{\partial \mathrm{L}}{\partial z^a} - \frac{d}{dx^i} \frac{\partial \mathrm{L}}{\partial z^a_i} = 0. \qquad (9)$$

We now introduce the 4-component of the tensor density P^i_α as canonically conjugate variables,

$$P_\alpha \equiv P^4_\alpha = \frac{\partial L}{\partial z^\alpha_4}. \qquad (10)$$

If we define further the Hamiltonian by

$$H(z^\alpha, z^\alpha_r, P_\alpha) = L - P_\alpha z^\alpha_4, \qquad (11)$$

the field equations (9) can be written in the canonical form

$$\left.\begin{aligned} \frac{\partial H}{\partial z^\alpha} - \frac{d}{dx^r}\frac{\partial H}{\partial z^\alpha_r} &= \frac{dP_\alpha}{dx^4}, \\ \frac{\partial H}{\partial P_\alpha} &= -\frac{dz^\alpha}{dx^4}. \end{aligned}\right\} \qquad (12)$$

(Throughout this paper indices i, k, l go from 1 to 4, indices r, s, t from 1 to 3, whereas indices α, β, γ go from 1 to $4n$, where n is the rank of the tensor z^α.)

The simplest proof of (12) consists in showing that the following relations hold:—

$$\frac{\partial H}{\partial z^\alpha_r} = \frac{\partial L}{\partial z^\alpha_r}; \qquad \frac{\partial H}{\partial z^\alpha} = \frac{\partial L}{\partial z^\alpha}, \qquad (13)$$

which we shall use later. They follow immediately from the definitions (10) and (11). For example,

$$\frac{\partial H}{\partial z^\alpha_r} = \frac{\partial L}{\partial z^\alpha_r} + \frac{\partial L}{\partial z^\beta_4}\frac{\partial z^\beta_4}{\partial z^\alpha_r} - P_\beta\frac{\partial z^\beta_4}{\partial z^\alpha_r} = \frac{\partial L}{\partial z^\alpha_r}.$$

This proof depends on the commutability of the factors in the Lagrangian, since differentiation with regard to a dependent variable is not allowed if the factors do not commute.

3. Quantization.

According to Heisenberg and Pauli the classical field equations are quantized by considering the field variables as non-commutable quantities, which satisfy the following commutation relations on a space section of constant time:—

$$\left.\begin{aligned} [P_\alpha(x), P_\beta(x')] &= [z^\alpha(x), z^\beta(x')] = 0 \\ [P_\alpha(x), z^\beta(x')] &= \delta^\beta_\alpha\delta(x^1 - x'^1)\delta(x^2 - x'^2)\delta(x^3 - x'^3) \\ &= \delta^\beta_\alpha\delta(x - x') \end{aligned}\right\} x^4 = x'^4. \qquad (14)$$

Here the bracket symbol is defined by

$$[\mathbf{A}, \mathbf{B}] = \frac{2\pi i}{h}(\mathbf{AB} - \mathbf{BA}).$$

We shall adopt the same commutation relations. Then we obtain, of course, all equations of the theory of Heisenberg and Pauli, some of which follow. By differentiating (14) we find

$$\left.\begin{aligned} &[z^\alpha(x),\ z^\beta_r(x')] = 0, \\ &[\mathrm{P}_\alpha(x),\ z^\beta_r(x')] = -\delta^\beta_\alpha \frac{d}{dx^r}\delta(x-x') = -\delta^\beta_\alpha\delta_r(x-x'). \end{aligned}\right\} \quad . \quad . \quad (15)$$

From (14) and (15) it follows in the usual way that for a functional $\mathbf{F}$ of the field variables

$$\left.\begin{aligned} &[z^\alpha, \mathbf{F}] = -\frac{\partial \mathrm{F}}{\partial \mathrm{P}_\alpha}; \qquad [\mathrm{P}_\alpha, \mathbf{F}] = \frac{\partial \mathrm{F}}{\partial z^\alpha} - \frac{d}{dx^r}\frac{\partial \mathrm{F}}{\partial z^\alpha_r}; \\ &\mathbf{F} = \int \mathrm{F}(z^\alpha, z^\alpha_r, \mathrm{P}_\alpha)dx^1dx^2dx^3. \end{aligned}\right\} \quad . \quad . \quad (16)$$

The integration over x^r in this equation extended over the whole of space has a meaning only if proper boundary conditions can be formulated. It is precisely for this reason that the x^r must be space-like co-ordinates and x^4 a time-like co-ordinate (*cf.* for example Weiss (1938)).

We introduce further the momentum operators. In the following I shall refer to H as the Hamiltonian density and to $\mathbf{H}$ as the Hamiltonian.

$$\left.\begin{aligned} &\mathrm{H}_r = -\mathrm{P}_\alpha z^\alpha_r \\ &\mathrm{H}_4 \equiv \mathrm{H} \end{aligned} \qquad \mathbf{H}_i = \int \mathrm{H}_i dx^1dx^2dx^3.\right\} \quad . \quad . \quad . \quad . \quad (17)$$

From (16) and the field equations (12) it then follows that

$$[\mathrm{P}_\alpha, \mathbf{H}_i] = \frac{d\mathrm{P}_\alpha}{dx^i}; \qquad [z^\alpha, \mathbf{H}_i] = \frac{dz^\alpha}{dx^i}. \quad . \quad . \quad . \quad . \quad (18)$$

By differentiating the second equation, we find

$$[z^\alpha_r, \mathbf{H}_i] = \frac{dz^\alpha_r}{dx^i}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

From (18) and (19) it follows that for a function F of the field variables

$$[\mathrm{F}, \mathbf{H}_i] = \frac{d\mathrm{F}}{dx^i}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

The equations (16) obviously remain correct if the function F depends explicitly on the co-ordinates x^i. Equation (20), on the other hand, does not hold in this case, but must be replaced by

$$[\mathrm{F}, \mathbf{H}_i] = \frac{d\mathrm{F}}{dx^i} - \frac{\partial \mathrm{F}}{\partial x^i}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

4. The Invariance of the Commutation Relations.

Consider an arbitrary infinitesimal transformation

$$\bar{x}^i = x^i + \epsilon\eta^i(x). \quad . \quad . \quad . \quad . \quad (22)$$

The z^a then transform according to an equation of the form

$$\bar{z}^a - z^a = \delta z^a = \epsilon a^a_\beta z^\beta, \quad . \quad . \quad . \quad . \quad (23)$$

where the a^a_β are given by the derivatives of the η^i, which we shall denote by

$$\frac{d\eta^i}{dx^k} = \eta^i_k. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

The ratio of the metric determinants in the two systems can be obtained from the Jacobian

$$\frac{\sqrt{(-\bar{g})}}{\sqrt{(-g)}} = \left| \frac{dx^i}{d\bar{x}^k} \right| = | \delta^i_k - \epsilon\eta^i_k | = 1 - \epsilon\eta^i_i. \quad . \quad . \quad . \quad (25)$$

With (8) the transformation of the P_a is therefore given by

$$\bar{P}_a - P_a = \delta P_a = -\epsilon a^\beta_a P_\beta + \epsilon\eta^4_i P^i_a - \epsilon\eta^i_i P_a. \quad . \quad . \quad (26)$$

Heisenberg and Pauli (1929) pointed out that there is a certain ambiguity in the definition of the P^r_a. According to (7) and (13) the following relation holds:—

$$P^r_a = \frac{\partial L}{\partial z^a_r} = \frac{\partial H}{\partial z^a_r}.$$

In quantum dynamics, as pointed out before, this relation is in general not correct. However, this does not mean that we must restrict ourselves to those simple cases in which this relation holds. For there is a corresponding ambiguity in the transition from the Euler equations to the Hamiltonian equations and in the choice of the sequence of factors in the Hamiltonian. But once we have decided on a certain sequence of factors in one co-ordinate system, the sequence in all other co-ordinate systems is determined. The ambiguity is reintroduced only if we revert to the classical definition of the P^r_a by means of the Lagrangian. In a consequent quantum theory we cannot return to the Euler equations and the Lagrangian. The correct definition to be used in quantum dynamics is that which uses the Hamiltonian density,

$$P^r_a = \frac{\partial H}{\partial z^a_r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Consider now two points $p(x^i)$ and $p'(x'^i)$, for which $x^4 = x'^4$. For

these two points the commutation relations (14) hold in the old co-ordinate system. But they do not necessarily hold in the new co-ordinate system, since the P_a are not tensor densities. However, in order that the commutation relations should be invariant, they must hold in the new co-ordinate system for two different points $\bar{p}$, $\bar{p}'$, which in the new system lie on a section $\bar{x}^4=$constant. In particular they must hold for the two points which in the new system have the same co-ordinates x^i, x'^i as the points p, p' in the old system. According to (22) these two points have in the old co-ordinate system the co-ordinates

$$\bar{p}(x^i-\epsilon\eta^i); \qquad \bar{p}'(x'^i-\epsilon\eta'^i). \qquad (28)$$

The condition that the commutation relations hold in the new co-ordinate system for the two points $\bar{p}$, $\bar{p}'$ is obviously a necessary and sufficient condition for the invariance of the commutation relations. We shall show that it is possible to find for any given value of x^4 a transformation $\mathbf{S}$, which transforms the field variables at the points p, p' in the old co-ordinate system, into the field variables at the points $\bar{p}$, $\bar{p}'$ in the new co-ordinate system, according to the formula

$$(\bar{P}_a)_{\bar{p}}=\mathbf{S}(P_a)_p\mathbf{S}^{-1}; \qquad (\bar{z}^a)_{\bar{p}}=\mathbf{S}(z^a)_p\mathbf{S}^{-1}, \qquad (29)$$

where the transformation $\mathbf{S}$ does not depend on the "space" co-ordinates x^r. A transformation of this type does not change the commutation brackets, since the δ-functions are c-numbers. It follows therefore that, if the commutation relations hold in the space $x^4=$constant in the old system, they also hold in the space $\bar{x}^4=$constant in the new system. They are therefore invariant.

It remains to show that a transformation $\mathbf{S}$ which satisfies (29) exists. We may write this equation in the form

$$\left.\begin{aligned}(\bar{P}_a)_{\bar{p}}-(P_a)_p=\overline{\delta P}_a=-\epsilon[P_a, \mathbf{T}],\\ (\bar{z}^a)_{\bar{p}}-(z^a)_p=\overline{\delta z}^a=-\epsilon[z^a, \mathbf{T}],\end{aligned}\right\} \qquad (30)$$

where $\mathbf{T}$ is defined by the equation

$$\mathbf{S}=1+\epsilon\mathbf{T}. \qquad (31)$$

The values of P_a, z^a at the points $\bar{p}$, $\bar{p}'$ in the old co-ordinate system are given by (compare (28))

$$(P_a)_{\bar{p}}=(P_a)_p-\epsilon\frac{dP_a}{dx^i}\eta^i,$$

$$(z^a)_{\bar{p}}=(z^a)_p-\epsilon\frac{dz^a}{dx^i}\eta^i.$$

If now we transform to the new co-ordinate system, the transformation formulæ (23), (26) give

$$\left.\begin{aligned}\overline{\delta}\mathrm{P}_\alpha &= -\epsilon a_\alpha^\beta \mathrm{P}_\beta - \epsilon\frac{d\mathrm{P}_\alpha}{dx^i}\eta^i + \epsilon\eta_r^4\mathrm{P}_\alpha^r - \epsilon\eta_r^r\mathrm{P}_\alpha,\\ \overline{\delta}z^\alpha &= \epsilon a_\beta^\alpha z^\beta - \epsilon\frac{dz^\alpha}{dx^i}\eta^i.\end{aligned}\right\} \qquad (32)$$

Comparing (32) with (30), it follows that $\mathbf{T}$ must satisfy the equations

$$\left.\begin{aligned}[z^\alpha, \mathbf{T}] &= -a_\beta^\alpha z^\beta + \frac{dz^\alpha}{dx^i}\eta^i,\\ [\mathrm{P}_\alpha, \mathbf{T}] &= a_\alpha^\beta \mathrm{P}_\beta + \frac{d\mathrm{P}_\alpha}{dx^i}\eta^i - \eta_r^4\frac{\partial \mathrm{H}}{\partial z_r^\alpha} + \eta_r^r\mathrm{P}_\alpha.\end{aligned}\right\} \qquad (33)$$

Here we have substituted for P_α^r, according to (27).

The following function satisfies the equations (33),

$$\left.\begin{aligned}\mathbf{T} &= \int \mathrm{T}\,dx^1dx^2dx^3,\\ \mathrm{T} &= \mathrm{P}_\alpha a_\beta^\alpha z^\beta + \mathrm{H}_i\eta^i.\end{aligned}\right\} \qquad (34)$$

Applying the equations (16), it is obvious that the first term in T yields the first term on the right-hand side of (33). The second term in T commuted with z^α yields

$$-\frac{\partial \mathrm{H}_i}{\partial \mathrm{P}_\alpha}\eta^i = [z^\alpha, \mathbf{H}_i]\eta^i = \frac{dz^\alpha}{dx^i}\eta^i$$

according to (18). The first equation (33) is therefore satisfied. For the commutation of the second term in T with P_α, equation (16) gives

$$\frac{\partial \mathrm{H}_i}{\partial z^\alpha}\eta^i - \frac{d}{dx^r}\left(\frac{\partial \mathrm{H}_i}{\partial z_r^\alpha}\eta^i\right) = \left(\frac{\partial \mathrm{H}_i}{\partial z^\alpha} - \frac{d}{dx^r}\frac{\partial \mathrm{H}_i}{\partial z_r^\alpha}\right)\eta^i - \frac{\partial \mathrm{H}_i}{\partial z_r^\alpha}\eta_r^i = [\mathrm{P}_\alpha, \mathbf{H}_i]\eta^i - \eta_r^4\frac{\partial \mathrm{H}_4}{\partial z_r^\alpha} - \eta_r^s\frac{\partial \mathrm{H}_s}{\partial z_r^\alpha}.$$

With (17) and (18) this reduces to

$$\frac{d\mathrm{P}_\alpha}{dx^i}\eta^i - \eta_r^4\frac{\partial \mathrm{H}}{\partial z_r^\alpha} + \eta_r^r\mathrm{P}_\alpha.$$

Thus the second equation (33) is also satisfied and the invariance of the commutation relations is proved.

5. The Transformation of the Hamiltonian.

The transformation $\mathbf{S}$ defined in the last paragraph can also be used to find a simple expression for the Hamiltonian in the new co-ordinate system. We take the field equations in the form (18)

$$[\mathrm{P}_\alpha, \mathbf{H}] = \frac{d\mathrm{P}_\alpha}{dx^4}; \qquad [z^\alpha, \mathbf{H}] = \frac{dz^\alpha}{dx^4}. \qquad (35)$$

Applying the transformation $\mathbf{S}$, we get

$$[\mathbf{S}\mathrm{P}_a\mathbf{S}^{-1}, \mathbf{SHS}^{-1}] = \mathbf{S}\frac{d\mathrm{P}_a}{dx^4}\mathbf{S}^{-1} = \frac{d}{dx^4}(\mathbf{S}\mathrm{P}_a\mathbf{S}^{-1}) - \frac{d\mathbf{S}}{dx^4}\mathrm{P}_a\mathbf{S}^{-1} - \mathbf{S}\mathrm{P}_a\frac{d\mathbf{S}^{-1}}{dx^4}$$

and a corresponding formula for z^a. According to (29) $\mathbf{S}\mathrm{P}_a\mathbf{S}^{-1}$ is the field variable in the new co-ordinate system at the point $\bar{p}$. Putting $\mathbf{S} = 1 + \epsilon\mathbf{T}$, we thus obtain

$$[(\bar{\mathrm{P}}_a)_{\bar{p}}, \mathbf{H} + \epsilon[\mathbf{T}, \mathbf{H}]] = \frac{d}{dx^4}(\bar{\mathrm{P}}_a)_{\bar{p}} + \epsilon\left[\mathrm{P}_a, \frac{d\mathbf{T}}{dx^4}\right].$$

The value of $[\mathbf{T}, \mathbf{H}]$ can be obtained from (21) by integration over the co-ordinates x^r, with the result

$$\left[(\bar{\mathrm{P}}_a)_{\bar{p}}, \mathbf{H} + \epsilon\left(\frac{d\mathbf{T}}{dx^4} - \frac{\partial\mathbf{T}}{\partial x^4}\right)\right] = \frac{d}{dx^4}(\bar{\mathrm{P}}_a)_{\bar{p}} + \epsilon\left[\mathrm{P}_a, \frac{d\mathbf{T}}{dx^4}\right].$$

The terms containing $d\mathbf{T}/dx^4$ cancel to the first order, and the equations of motion in the new co-ordinate may, therefore, be written in the form

$$[(\bar{\mathrm{P}}_a)_{\bar{p}}, \bar{\mathbf{H}}_{\bar{p}}] = \left(\frac{d\bar{\mathrm{P}}_a}{d\bar{x}^4}\right)_{\bar{p}}; \qquad [(\bar{z}^a)_{\bar{p}}, \bar{\mathbf{H}}_{\bar{p}}] = \left(\frac{d\bar{z}^a}{d\bar{x}^4}\right)_{\bar{p}}; \qquad \bar{\mathbf{H}}_{\bar{p}} = \mathbf{H} - \epsilon\frac{\partial\mathbf{T}}{\partial x^4}. \tag{36}$$

Here we have used the formula

$$\frac{d}{dx^4}(\bar{\mathrm{P}}_a)_{\bar{p}} = \left(\frac{d\bar{\mathrm{P}}_a}{d\bar{x}^4}\right)_{\bar{p}}, \qquad \frac{d}{dx^4}(\bar{z}^a)_{\bar{p}} = \left(\frac{d\bar{z}^a}{d\bar{x}^4}\right)_{\bar{p}}. \tag{37}$$

The latter equation may also be written in the form

$$\overline{\delta z}^a_4 = (\bar{z}^a_4)_{\bar{p}} - (z^a_4)_p = \frac{d}{dx^4}((\bar{z}^a)_{\bar{p}} - (z^a)_p) = \frac{d}{dx^4}\overline{\delta z}^a. \tag{37a}$$

(37) can easily be proved, for we have

$$\frac{d\bar{\mathrm{P}}_a}{d\bar{x}^4} = \frac{d\bar{\mathrm{P}}_a}{dx^4} - \epsilon\frac{d\bar{\mathrm{P}}_a}{dx^i}\eta^i_4.$$

Therefore the right-hand side of (37) is to the first order

$$\begin{aligned}\left(\frac{d\bar{\mathrm{P}}_a}{d\bar{x}^4}\right)_{\bar{p}} &= \left(\frac{d\bar{\mathrm{P}}_a}{d\bar{x}^4}\right)_p - \epsilon\frac{d}{dx^i}\left(\frac{d\bar{\mathrm{P}}_a}{d\bar{x}^4}\right)\eta^i \\ &= \frac{d\bar{\mathrm{P}}_a}{dx^4} - \epsilon\frac{d\bar{\mathrm{P}}_a}{dx^i}\eta^i_4 - \epsilon\frac{d^2\bar{\mathrm{P}}_a}{dx^4dx^i}\eta^i \\ &= \frac{d}{dx^4}\left(\bar{\mathrm{P}}_a - \epsilon\frac{d\bar{\mathrm{P}}_a}{dx^i}\eta^i\right) = \frac{d}{dx^4}(\bar{\mathrm{P}}_a)_{\bar{p}}.\end{aligned}$$

The equation (36) is exactly the same as the corresponding equation in classical mechanics (*cf.* for example Nordheim and Fues (1927)). It has

been shown by Jordan (1926) that the equations (29) for the transformation of the field variables can also be written in a form analogous to the corresponding classical equations. It is interesting that the analogy apparently also extends to the case when the transformation **S** depends explicitly on the time.

6. Classical Transformation of the Hamiltonian.

In the last paragraph we determined the transformation of the Hamiltonian in such a way that the equations of motion are invariant. The law of transformation coincided with the classical law in the limiting case $h=0$. If this were not the case, the results of quantization would depend on the co-ordinate system in which the quantization has been carried out. However, the transformation of the Hamiltonian is not completely determined by the argument of the last paragraph. In the classical theory any complete differential may be added to the Hamiltonian density.

Indeed, if we transform the Hamiltonian density explicitly, we obtain a result differing from (36) by a complete differential. This can easily be shown. The Hamiltonian density in the classical theory is defined by

$$\mathrm{H}=\mathrm{L}-\mathrm{P}_\alpha z_4^\alpha, \qquad (38)$$

where L is a scalar density. If we transform to another co-ordinate system, the change in the Hamiltonian density is given by (compare (25))

$$\delta\mathrm{H}=-\epsilon\eta_i^i\mathrm{L}-\delta\mathrm{P}_\alpha z_4^\alpha-\mathrm{P}_\alpha\delta z_4^\alpha.$$

For the Hamiltonian density at the point $\bar{p}$ we find

$$\overline{\delta\mathrm{H}}=\bar{\mathrm{H}}_{\bar{p}}-\mathrm{H}_p=-\epsilon\eta_i^i\mathrm{L}-\epsilon\frac{d\mathrm{L}}{dx^i}\eta^i-\overline{\delta\mathrm{P}}_\alpha z_4^\alpha-\mathrm{P}_\alpha\overline{\delta z_4^\alpha}.$$

Reintroducing H for L, this may be written in the form

$$\overline{\delta\mathrm{H}}=-\epsilon\eta_i^i(\mathrm{H}+\mathrm{P}_\alpha z_4^\alpha)-\epsilon\frac{d(\mathrm{H}+\mathrm{P}_\alpha z_4^\alpha)}{dx^i}\eta^i-\overline{\delta\mathrm{P}}_\alpha z_4^\alpha-\mathrm{P}_\alpha\overline{\delta z_4^\alpha}.$$

We introduce the values (32), (37*a*) for $\overline{\delta\mathrm{P}}_\alpha$ and $\overline{\delta z_4^\alpha}$. A number of terms cancel and we get easily

$$\overline{\delta\mathrm{H}}=-\epsilon\frac{d}{dx^i}(\eta^i\mathrm{H})-\epsilon\eta_i^4\mathrm{P}_\alpha^i z_4^\alpha-\epsilon\mathrm{P}_\alpha\frac{da_\beta^\alpha}{dx^4}z^\beta+\epsilon\mathrm{P}_\alpha z_i^\alpha\eta_4^i.$$

We now separate all terms with $i=4$ and carry through the differentiation of H with regard to x^4:

$$\overline{\delta\mathrm{H}}=-\epsilon\frac{\partial}{\partial x^4}(\eta^4\mathrm{H}+\mathrm{P}_\alpha a_\beta^\alpha z^\beta-\mathrm{P}_\alpha z_r^\alpha\eta^r)-\epsilon\frac{d}{dx^r}(\eta^r\mathrm{H})$$
$$-\epsilon\eta^4\left(\frac{\partial\mathrm{H}}{\partial z^\alpha}z_4^\alpha+\frac{\partial\mathrm{H}}{\partial z_r^\alpha}\frac{dz_r^\alpha}{dx^4}+\frac{\partial\mathrm{H}}{\partial\mathrm{P}_\alpha}\frac{d\mathrm{P}_\alpha}{dx^4}\right)-\epsilon\eta_r^4\mathrm{P}_\alpha^r z_4^\alpha.$$

Applying the field equations (12) and the definition (17), the result is

$$\overline{\delta \mathrm{H}} = -\epsilon \frac{\partial}{\partial x^4}(\eta^i \mathrm{H}_i + \mathrm{P}_\alpha a^\alpha_\beta z^\beta) - \epsilon \frac{d}{dx^r}(\eta^r \mathrm{H}) - \epsilon \eta^4 \left(\frac{d\mathrm{P}^r_\alpha}{dx^r} z^\alpha_4 + \mathrm{P}^r_\alpha \frac{dz^\alpha_r}{dx^4} \right) - \epsilon \eta^4_r \mathrm{P}^r_\alpha z^\alpha_4$$

or

$$\overline{\delta \mathrm{H}} = -\epsilon \frac{\partial}{\partial x^4}(\eta^i \mathrm{H}_i + \mathrm{P}_\alpha a^\alpha_\beta z^\beta) - \epsilon \frac{d}{dx^r}(\eta^r \mathrm{H} + \eta^4 \mathrm{P}^r_\alpha z^\alpha_4), \quad . \quad . \quad (39)$$

which is identical with (36) apart from the last term.

7. The Complete Differential in the Hamiltonian Density.

If the boundary conditions of our problem are such that the Hamiltonian density vanishes sufficiently rapidly at the boundaries, the additional term in (39) does not matter. However, in many applications it is more convenient to assume periodicity at the boundaries. Therefore it is of interest to show in general that the addition of a term of the form

$$\int \frac{d}{dx^r}\{\mathrm{F}(z^\alpha, z^\alpha_r, \mathrm{P}_\alpha, x^i)\} d\tau, \qquad d\tau = dx^1 dx^2 dx^3, \quad . \quad . \quad (40)$$

to the Hamiltonian does not change the equations of motion. With other words:

$$\left[z^\alpha, \int \frac{d}{dx^r} \mathrm{F} d\tau \right] = \left[\mathrm{P}_\alpha, \int \frac{d}{dx^r} \mathrm{F} d\tau \right] = 0. \quad . \quad . \quad . \quad (41)$$

With (21) this may be written in the form

$$\left. \begin{aligned} [z^\alpha [\mathbf{F}, \mathbf{H}_r]] + \left[z^\alpha, \int \frac{\partial}{\partial x^r} \mathrm{F} d\tau \right] &= 0, \\ [\mathrm{P}_\alpha, [\mathbf{F}, \mathbf{H}_r]] + \left[\mathrm{P}_\alpha, \int \frac{\partial}{\partial x^r} \mathrm{F} d\tau \right] &= 0. \end{aligned} \right\} \quad . \quad . \quad . \quad (42)$$

We use the equation

$$[\mathbf{A}, [\mathbf{B}, \mathbf{C}]] = [\mathbf{B}, [\mathbf{A}, \mathbf{C}]] + [[\mathbf{A}, \mathbf{B}], \mathbf{C}],$$

and obtain for (42)

$$[\mathbf{F}, [z^\alpha, \mathbf{H}_r]] + [[z^\alpha, \mathbf{F}], \mathbf{H}_r] + \left[z^\alpha, \int \frac{\partial}{\partial x^r} \mathrm{F} d\tau \right] = 0,$$

$$[\mathbf{F}, [\mathrm{P}_\alpha, \mathbf{H}_r]] + [[\mathrm{P}_\alpha, \mathbf{F}], \mathbf{H}_r] + \left[\mathrm{P}_\alpha, \int \frac{\partial}{\partial x^r} \mathrm{F} d\tau \right] = 0,$$

or with (16) and (18)

$$\left. \begin{aligned} [\mathbf{F}, z^\alpha_r] - \left[\frac{\partial \mathrm{F}}{\partial \mathrm{P}_\alpha}, \mathbf{H}_r \right] - \frac{\partial}{\partial \mathrm{P}_\alpha} \frac{\partial \mathrm{F}}{\partial x^r} &= 0, \\ \left[\mathbf{F}, \frac{d\mathrm{P}_\alpha}{dx^r} \right] + \left[\frac{\partial \mathrm{F}}{\partial z^\alpha} - \frac{d}{dx^s} \frac{\partial \mathrm{F}}{\partial z^\alpha_s}, \mathbf{H}_r \right] + \frac{\partial}{\partial z^\alpha} \frac{\partial \mathrm{F}}{\partial x^r} - \frac{d}{dx^s} \frac{\partial}{\partial z^\alpha_s} \frac{\partial \mathrm{F}}{\partial x^r} &= 0. \end{aligned} \right\} \quad . \quad (43)$$

Differentiating (16) we obtain

$$[z^a_r, \mathbf{F}] = -\frac{d}{dx^r}\frac{\partial F}{\partial P_a}; \qquad \left[\frac{dP_a}{dx^r}, \mathbf{F}\right] = \frac{d}{dx^r}\frac{\partial F}{\partial z^a} - \frac{d^2}{dx^r dx^s}\frac{\partial F}{\partial z^a_s}.$$

Equation (43) therefore yields, together with (21),

$$\frac{d}{dx^r}\frac{\partial F}{\partial P_a} - \left(\frac{d}{dx^r}\frac{\partial F}{\partial P_a} - \frac{\partial}{\partial x^r}\frac{\partial F}{\partial P_a}\right) - \frac{\partial}{\partial P_a}\frac{\partial F}{\partial x^r} = 0,$$

$$-\frac{d}{dx^r}\frac{\partial F}{\partial z^a} + \frac{d^2}{dx^r dx^s}\frac{\partial F}{\partial z^a_s} + \left(\frac{d}{dx^r} - \frac{\partial}{\partial x^r}\right)\left(\frac{\partial F}{\partial z^a} - \frac{d}{dx^s}\frac{\partial F}{\partial z^a_s}\right) + \frac{\partial}{\partial z^a}\frac{\partial F}{\partial x^r} - \frac{d}{dx^s}\frac{\partial}{\partial z^a_s}\frac{\partial F}{\partial x^r} = 0.$$

All partial differentiations can of course be exchanged. The first equation is therefore an identity. The second reduces to

$$\frac{\partial}{\partial x^r}\frac{d}{dx^s}\frac{\partial F}{\partial z^a_s} - \frac{d}{dx^s}\frac{\partial}{\partial x^r}\frac{\partial F}{\partial z^a_s} = 0.$$

We shall now show that $\partial/\partial x^r$ and d/dx^s can also be exchanged, so that this equation is an identity too. The derivative of a function

$$f(z^a, z^a_r, P_a, x^t) = f(Q_\rho, x^t)$$

(Q_ρ stands for the field variables z^a, P_a and the derivatives z^a_r) is defined by

$$\frac{d}{dx^s}f(Q_\rho, x^t) = \lim_{\lambda \to 0}\frac{f\left(Q_\rho + \lambda\frac{dQ_\rho}{dx^s}, x^t + \delta^t_s\lambda\right) - f(Q_\rho, x^t)}{\lambda}.$$

Therefore

$$\frac{\partial}{\partial x^r}\frac{d}{dx^s}f(Q_\rho, x^t) = \lim_{\lambda, \lambda' \to 0}\frac{1}{\lambda\lambda'}\left\{\begin{array}{l} f\left(Q_\rho + \lambda\frac{dQ_\rho}{dx^s} + \lambda\lambda'\frac{\partial}{\partial x^r}\frac{dQ_\rho}{dx^s}, x^t + \delta^t_s\lambda + \delta^t_r\lambda'\right) \\ -f(Q_\rho, x^t + \delta^t_r\lambda') \\ -f\left(Q_\rho + \lambda\frac{dQ_\rho}{dx^s}, x^t + \delta^t_s\lambda\right) + f(Q_\rho, x^t). \end{array}\right\} \tag{44}$$

On the other hand,

$$\frac{\partial}{\partial x^r}f(Q_\rho, x^t) = \lim_{\lambda' \to 0}\frac{f(Q_\rho, x^t + \delta^t_r\lambda') - f(Q_\rho, x^t)}{\lambda'},$$

and

$$\frac{d}{dx^s}\frac{\partial}{\partial x^r}f(Q_\rho, x^t) = \lim_{\lambda, \lambda' \to 0}\frac{1}{\lambda\lambda'}\left\{\begin{array}{l} f\left(Q_\rho + \lambda\frac{dQ_\rho}{dx^s}, x^t + \delta^t_r\lambda' + \delta^t_s\lambda\right) \\ -f\left(Q_\rho + \lambda\frac{dQ_\rho}{dx^s}, x^t + \delta^t_s\lambda\right) \\ f(Q_\rho, x^t + \delta^t_r\lambda') + f(Q_\rho, x^t). \end{array}\right\}. \tag{45}$$

This expression is identical with (44) provided

$$\frac{\partial}{\partial x^r}\frac{dQ_\rho}{dx^s}=0.$$

This coincides with the following three expressions:—

$$\frac{\partial}{\partial x^r}\frac{dz^a}{dx^s}=\frac{\partial}{\partial x^r}\frac{dP_a}{dx^s}=\frac{\partial}{\partial x^r}\frac{dz_i^a}{dx^s}=0.$$

These equations follow immediately from (18) and (19) by partial differentiation, since the left-hand side of these equations does not depend explicitly on the co-ordinates x^r.

Thus it is proved that the addition of a term of the form (40) does not change the equations of motion, and the proof of the invariance of the whole scheme of quantum dynamics is completed.

I wish to thank Professor M. Born for many stimulating discussions on the subject.

Summary.

The invariance of quantum dynamics in the form developed by Heisenberg and Pauli (1929) against all transformations allowed by the general theory of relativity is proved.

REFERENCES TO LITERATURE.

Heisenberg, W., and Pauli, W., 1929. "Zur Quantendynamik der Wellenfelder," *Zeits. Phys.*, vol. lvi, p. 1.

Infeld, L., 1937. "The Lorentz Transformations in the New Quantum Electrodynamics," *Proc. Roy. Soc.* (A), vol. clviii, p. 368.

Jordan, P., 1926. "Über kanonische Transformationen in der Quantenmechanik," *Zeits. Phys.*, vol. xxxvii, p. 383; vol. xxxviii, p. 513.

Nordheim, L., and Fues, E., 1927. "Die Hamilton-Jakobische Theorie der Dynamik," *Handbuch der Physik*, vol. v, p. 91 (see particularly equation (5), p. 98).

Pryce, M. H. L., 1937. "On the New Field Theory," *Proc. Roy. Soc.* (A), vol. clix, p. 355.

Weiss, P., 1938. "On the Hamilton-Jacobi Theory and Quantization of Generalised Electrodynamics," *Proc. Roy. Soc.* (A), vol. clxix, p. 102.

(*Issued separately May* 18, 1939.)

XI.—**The Relation between Mathematics and Physics.** By **Professor P. A. M. Dirac,** F.R.S. *Communicated to the* Royal Society of Edinburgh on presentation of the JAMES SCOTT Prize, February 6, 1939.

(MS. received February 25, 1939.)

THE physicist, in his study of natural phenomena, has two methods of making progress: (1) the method of experiment and observation, and (2) the method of mathematical reasoning. The former is just the collection of selected data; the latter enables one to infer results about experiments that have not been performed. There is no logical reason why the second method should be possible at all, but one has found in practice that it does work and meets with remarkable success. This must be ascribed to some *mathematical quality in Nature*, a quality which the casual observer of Nature would not suspect, but which nevertheless plays an important rôle in Nature's scheme.

One might describe the mathematical quality in Nature by saying that the universe is so constituted that mathematics is a useful tool in its description. However, recent advances in physical science show that this statement of the case is too trivial. The connection between mathematics and the description of the universe goes far deeper than this, and one can get an appreciation of it only from a thorough examination of the various factors that make it up. The main aim of my talk to you will be to give you such an appreciation. I propose to deal with how the physicist's views on this subject have been gradually modified by the succession of recent developments in physics, and then I would like to make a little speculation about the future.

Let us take as our starting-point that scheme of physical science which was generally accepted in the last century—the mechanistic scheme. This considers the whole universe to be a dynamical system (of course an extremely complicated dynamical system), subject to laws of motion which are essentially of the Newtonian type. The rôle of mathematics in this scheme is to represent the laws of motion by equations, and to obtain solutions of the equations referring to observed conditions.

The dominating idea in this application of mathematics to physics is that the equations representing the laws of motion *should be of a simple form*. The whole success of the scheme is due to the fact that equations of simple form do seem to work. The physicist is thus provided with a

principle of simplicity, which he can use as an instrument of research. If he obtains, from some rough experiments, data which fit in roughly with certain simple equations, he infers that if he performed the experiments more accurately he would obtain data fitting in more accurately with the equations. The method is much restricted, however, since the principle of simplicity applies only to fundamental laws of motion, not to natural phenomena in general. For example, rough experiments about the relation between the pressure and volume of a gas at a fixed temperature give results fitting in with a law of inverse proportionality, but it would be wrong to infer that more accurate experiments would confirm this law with greater accuracy, as one is here dealing with a phenomenon which is not connected in any very direct way with the fundamental laws of motion.

The discovery of the theory of relativity made it necessary to modify the principle of simplicity. Presumably one of the fundamental laws of motion is the law of gravitation which, according to Newton, is represented by a very simple equation, but, according to Einstein, needs the development of an elaborate technique before its equation can even be written down. It is true that, from the standpoint of higher mathematics, one can give reasons in favour of the view that Einstein's law of gravitation is actually simpler than Newton's, but this involves assigning a rather subtle meaning to simplicity, which largely spoils the practical value of the principle of simplicity as an instrument of research into the foundations of physics.

What makes the theory of relativity so acceptable to physicists in spite of its going against the principle of simplicity is its great *mathematical beauty*. This is a quality which cannot be defined, any more than beauty in art can be defined, but which people who study mathematics usually have no difficulty in appreciating. The theory of relativity introduced mathematical beauty to an unprecedented extent into the description of Nature. The restricted theory changed our ideas of space and time in a way that may be summarised by stating that the group of transformations to which the space-time continuum is subject must be changed from the Galilean group to the Lorentz group. The latter group is a much more beautiful thing than the former—in fact, the former would be called mathematically a degenerate special case of the latter. The general theory of relativity involved another step of a rather similar character, although the increase in beauty this time is usually considered to be not quite so great as with the restricted theory, which results in the general theory being not quite so firmly believed in as the restricted theory.

We now see that we have to change the principle of simplicity into a

principle of mathematical beauty. The research worker, in his efforts to express the fundamental laws of Nature in mathematical form, should strive mainly for mathematical beauty. He should still take simplicity into consideration in a subordinate way to beauty. (For example Einstein, in choosing a law of gravitation, took the simplest one compatible with his space-time continuum, and was successful.) It often happens that the requirements of simplicity and of beauty are the same, but where they clash the latter must take precedence.

Let us pass on to the second revolution in physical thought of the present century—the quantum theory. This is a theory of atomic phenomena based on a mechanics of an essentially different type from Newton's. The difference may be expressed concisely, but in a rather abstract way, by saying that dynamical variables in quantum mechanics are subject to an algebra in which the commutative axiom of multiplication does not hold. Apart from this, there is an extremely close formal analogy between quantum mechanics and the old mechanics. In fact, it is remarkable how adaptable the old mechanics is to the generalization of non-commutative algebra. All the elegant features of the old mechanics can be carried over to the new mechanics, where they reappear with an enhanced beauty.

Quantum mechanics requires the introduction into physical theory of a vast new domain of pure mathematics—the whole domain connected with non-commutative multiplication. This, coming on top of the introduction of new geometries by the theory of relativity, indicates a trend which we may expect to continue. We may expect that in the future further big domains of pure mathematics will have to be brought in to deal with the advances in fundamental physics.

Pure mathematics and physics are becoming ever more closely connected, though their methods remain different. One may describe the situation by saying that the mathematician plays a game in which he himself invents the rules while the physicist plays a game in which the rules are provided by Nature, but as time goes on it becomes increasingly evident that the rules which the mathematician finds interesting are the same as those which Nature has chosen. It is difficult to predict what the result of all this will be. Possibly, the two subjects will ultimately unify, every branch of pure mathematics then having its physical application, its importance in physics being proportional to its interest in mathematics. At present we are, of course, very far from this stage, even with regard to some of the most elementary questions. For example, only four-dimensional space is of importance in physics, while spaces with other numbers of dimensions are of about equal interest in mathematics.

It may well be, however, that this discrepancy is due to the incompleteness of present-day knowledge, and that future developments will show four-dimensional space to be of far greater mathematical interest than all the others.

The trend of mathematics and physics towards unification provides the physicist with a powerful new method of research into the foundations of his subject, a method which has not yet been applied successfully, but which I feel confident will prove its value in the future. The method is to begin by choosing that branch of mathematics which one thinks will form the basis of the new theory. One should be influenced very much in this choice by considerations of mathematical beauty. It would probably be a good thing also to give a preference to those branches of mathematics that have an interesting group of transformations underlying them, since transformations play an important rôle in modern physical theory, both relativity and quantum theory seeming to show that transformations are of more fundamental importance than equations. Having decided on the branch of mathematics, one should proceed to develop it along suitable lines, at the same time looking for that way in which it appears to lend itself naturally to physical interpretation.

This method was used by Jordan in an attempt to get an improved quantum theory on the basis of an algebra with non-associative multiplication. The attempt was not successful, as one would rather expect, if one considers that non-associative algebra is not a specially beautiful branch of mathematics, and is not connected with an interesting transformation theory. I would suggest, as a more hopeful-looking idea for getting an improved quantum theory, that one take as basis the theory of functions of a complex variable. This branch of mathematics is of exceptional beauty, and further, the group of transformations with which it is connected, namely, the group of transformations in the complex plane, is the same as the Lorentz group governing the space-time of restricted relativity. One is thus led to suspect the existence of some deep-lying connection between the theory of functions of a complex variable and the space-time of restricted relativity, the working out of which will be a difficult task for the future.

Let us now discuss the extent of the mathematical quality in Nature. According to the mechanistic scheme of physics or to its relativistic modification, one needs for the complete description of the universe not merely a complete system of equations of motion, but also a complete set of initial conditions, and it is only to the former of these that mathematical theories apply. The latter are considered to be not amenable to theoretical treatment and to be determinable only from observation.

The enormous complexity of the universe is ascribed to an enormous complexity in the initial conditions, which removes them beyond the range of mathematical discussion.

I find this position very unsatisfactory philosophically, as it goes against all ideas of the *unity of Nature*. Anyhow, if it is only to a part of the description of the universe that mathematical theory applies, this part ought certainly to be sharply distinguished from the remainder. But in fact there does not seem to be any natural place in which to draw the line. Are such things as the properties of the elementary particles of physics, their masses and the numerical coefficients occurring in their laws of force, subject to mathematical theory? According to the narrow mechanistic view, they should be counted as initial conditions and outside mathematical theory. However, since the elementary particles all belong to one or other of a number of definite types, the members of one type being all exactly similar, they must be governed by mathematical law to some extent, and most physicists now consider it to be quite a large extent. For example, Eddington has been building up a theory to account for the masses. But even if one supposed all the properties of the elementary particles to be determinable by theory, one would still not know where to draw the line, as one would be faced by the next question—Are the relative abundances of the various chemical elements determinable by theory? One would pass gradually from atomic to astronomic questions.

This unsatisfactory situation gets changed for the worse by the new quantum mechanics. In spite of the great analogy between quantum mechanics and the older mechanics with regard to their mathematical formalisms, they differ drastically with regard to the nature of their physical consequences. According to the older mechanics, the result of any observation is determinate and can be calculated theoretically from given initial conditions; but with quantum mechanics there is usually an indeterminacy in the result of an observation, connected with the possibility of occurrence of a quantum jump, and the most that can be calculated theoretically is the probability of any particular result being obtained. The question, which particular result will be obtained in some particular case, lies outside the theory. This must not be attributed to an incompleteness of the theory, but is essential for the application of a formalism of the kind used by quantum mechanics.

Thus according to quantum mechanics we need, for a complete description of the universe, not only the laws of motion and the initial conditions, but also information about which quantum jump occurs in each case when a quantum jump does occur. The latter information

must be included, together with the initial conditions, in that part of the description of the universe outside mathematical theory.

The increase thus arising in the non-mathematical part of the description of the universe provides a philosophical objection to quantum mechanics, and is, I believe, the underlying reason why some physicists still find it difficult to accept this mechanics. Quantum mechanics should not be abandoned, however, firstly, because of its very widespread and detailed agreement with experiment, and secondly, because the indeterminacy it introduces into the results of observations is of a kind which is philosophically satisfying, being readily ascribable to an inescapable crudeness in the means of observation available for small-scale experiments. The objection does show, all the same, that the foundations of physics are still far from their final form.

We come now to the third great development of physical science of the present century—the new cosmology. This will probably turn out to be philosophically even more revolutionary than relativity or the quantum theory, although at present one can hardly realize its full implications. The starting-point is the observed red-shift in the spectra of distant heavenly bodies, indicating that they are receding from us with velocities proportional to their distances.* The velocities of the more distant ones are so enormous that it is evident we have here a fact of the utmost importance, not a temporary or local condition, but something fundamental for our picture of the universe.

If we go backwards into the past we come to a time, about 2×10^9 years ago, when all the matter in the universe was concentrated in a very small volume. It seems as though something like an explosion then took place, the fragments of which we now observe still scattering outwards. This picture has been elaborated by Lemaître, who considers the universe to have started as a single very heavy atom, which underwent violent radioactive disintegrations and so broke up into the present collection of astronomical bodies, at the same time giving off the cosmic rays.

With this kind of cosmological picture one is led to suppose that there was a beginning of time, and that it is meaningless to inquire into what happened before then. One can get a rough idea of the geometrical relationships this involves by imagining the present to be the surface of a

* The recession velocities are not strictly proved, since one may postulate some other cause for the spectral red-shift. However, the new cause would presumably be equally drastic in its effect on cosmological theory and would still need the introduction of a parameter of the order 2×10^9 years for its mathematical discussion, so it would probably not disturb the essential ideas of the argument in the text.

sphere, going into the past to be going in towards the centre of the sphere, and going into the future to be going outwards. There is then no limit to how far one may go into the future, but there is a limit to how far one can go into the past, corresponding to when one has reached the centre of the sphere. The beginning of time provides a natural origin from which to measure the time of any event. The result is usually called the epoch of that event. Thus the present epoch is 2×10^9 years.

Let us now return to dynamical questions. With the new cosmology the universe must have been started off in some very simple way. What, then, becomes of the initial conditions required by dynamical theory? Plainly there cannot be any, or they must be trivial. We are left in a situation which would be untenable with the old mechanics. If the universe were simply the motion which follows from a given scheme of equations of motion with trivial initial conditions, it could not contain the complexity we observe. Quantum mechanics provides an escape from the difficulty. It enables us to ascribe the complexity to the quantum jumps, lying outside the scheme of equations of motion. *The quantum jumps now form the uncalculable part of natural phenomena, to replace the initial conditions of the old mechanistic view.*

One further point in connection with the new cosmology is worthy of note. At the beginning of time the laws of Nature were probably very different from what they are now. Thus we should consider the laws of Nature as continually changing with the epoch, instead of as holding uniformly throughout space-time. This idea was first put forward by Milne, who worked it out on the assumptions that the universe at a given epoch is roughly everywhere uniform and spherically symmetrical. I find these assumptions not very satisfying, because the local departures from uniformity are so great and are of such essential importance for our world of life that it seems unlikely there should be a principle of uniformity overlying them. Further, as we already have the laws of Nature depending on the epoch, we should expect them also to depend on position in space, in order to preserve the beautiful idea of the theory of relativity that there is fundamental similarity between space and time. This goes more drastically against Milne's assumptions than a mere lack of uniformity in the distribution of matter.

•

We have followed through the main course of the development of the relation between mathematics and physics up to the present time, and have reached a stage where it becomes interesting to indulge in speculations about the future. There has always been an unsatisfactory feature in the relation, namely, the limitation in the extent to which

mathematical theory applies to a description of the physical universe. The part to which it does not apply has suffered an increase with the arrival of quantum mechanics and a decrease with the arrival of the new cosmology, but has always remained.

This feature is so unsatisfactory that I think it safe to predict it will disappear in the future, in spite of the startling changes in our ordinary ideas to which we should then be led. It would mean the existence of a scheme in which the whole of the description of the universe has its mathematical counterpart, and we must suppose that a person with a complete knowledge of mathematics could deduce, not only astronomical data, but also all the historical events that take place in the world, even the most trivial ones. Of course, it must be beyond human power actually to make these deductions, since life as we know it would be impossible if one could calculate future events, but the methods of making them would have to be well defined. The scheme could not be subject to the principle of simplicity since it would have to be extremely complicated, but it may well be subject to the principle of mathematical beauty.

I would like to put forward a suggestion as to how such a scheme might be realized. If we express the present epoch, 2×10^9 years, in terms of a unit of time defined by the atomic constants, we get a number of the order 10^{39}, which characterizes the present in an absolute sense. Might it not be that all present events correspond to properties of this large number, and, more generally, that the whole history of the universe corresponds to properties of the whole sequence of natural numbers? At first sight it would seem that the universe is far too complex for such a correspondence to be possible. But I think this objection cannot be maintained, since a number of the order 10^{39} is *excessively* complicated, just because it is so enormous. We have a brief way of writing it down, but this should not blind us to the fact that it must have excessively complicated properties.

There is thus a possibility that the ancient dream of philosophers to connect all Nature with the properties of whole numbers will some day be realized. To do so physics will have to develop a long way to establish the details of how the correspondence is to be made. One hint for this development seems pretty obvious, namely, the study of whole numbers in modern mathematics is inextricably bound up with the theory of functions of a complex variable, which theory we have already seen has a good chance of forming the basis of the physics of the future. The working out of this idea would lead to a connection between atomic theory and cosmology.

(*Issued separately May* 20, 1939.)

XII.—The Molecular Spectra of the Hydrogen Isotopes. II.—The Assumption of a Common Potential Function for the Isotopic States.

By **Ian Sandeman,** D.Sc., late Carnegie Research Scholar in the University of St Andrews.

(MS. received February 2, 1939. Read May 1, 1939.)

In Part I (1938) an application of the theoretical work of the late J. L. Dunham (1932) to the $1s\sigma 2p\sigma {}^1\Sigma$ and $1s\sigma 2s\sigma {}^3\Sigma$ states of D_2 has been carried out, and the constants of these states have been calculated and compared with those for the corresponding states of H_2. This work shows that the corresponding isotopic states are much more dissimilar than we should have expected.

The dissimilarity consists of:

(1) A small difference in the equilibrium internuclear distance, this distance being appreciably greater for H_2 than for D_2 in the $1s\sigma 2p\sigma {}^1\Sigma$ state.

(2) A difference in the a_0 constants of the potential functions. (See equation (1) below. This means that the force of restitution, when the molecule is given a slight displacement from its equilibrium position, is not quite the same for the two isotopes. The actual effect is not similar for the two states considered, and not to be explained by the slight difference in the equilibrium distance.)

(3) A difference in the form of the potential function.

The last-mentioned is perhaps the most striking dissimilarity between the isotopes. When we express the potential function as a power series in terms of the internuclear separation (equation (2) of Part I):

$$U = a_0\xi^2(1 + a_1\xi + a_2\xi^2 + a_3\xi^3 + \ldots), \quad . \quad . \quad . \quad (1)$$

and calculate the values of the configuration constants, a_1, a_2, a_3, . . ., from the spectra, we find the following values for the $1s\sigma 2s\sigma {}^3\Sigma$ states of D_2 and H_2 (see Table IV of Part I):—

	D_2.	H_2.
a_1	−1·6509	−1·6431
a_2	2·0066	1·9565
a_3	−2·1545	−2·0897
a_4	1·9199	2·2681
a_5	−1·1920	−2·9779
a_6	0·4960	4·1706

A similar situation has been disclosed by the work of Crawford and Jorgensen (1936) on the band spectra of the isotopic molecules Li^7H and Li^7D. In the case of the molecules H_2 and D_2 the dissimilarity appears, however, to be more fundamental, since the D_2 series begins to bend back at a_4, while the H_2 series does not.

Jevons says on p. 209 of his "Report on Band-spectra of Diatomic Molecules" (1932): "The isotopes of a given element have the same net nuclear charge and the same number and configuration of extranuclear electrons, but different numbers and configurations of nuclear electrons and protons. This difference of nuclear constitution involves (i) a difference of nuclear mass, (ii) a difference of nuclear spin angular momentum, and (iii) a slight difference of extranuclear electric field. Isotopes are therefore identical in all chemical and physical properties but those which depend on any of these three factors." The order in which Jevons quotes the three factors is the order in which these factors are commonly thought to affect band spectra.

In the case of line spectra, apart from the isotope effect of the hyperfine structure arising from a difference of spin momentum of the isotopic nuclei, the only observed effect is an electronic effect on the translatory motion of the atomic nucleus. Owing to the fact that the electrons have some mass, the centre of mass of the atom does not coincide with the centre of mass of the atomic nucleus. Thus in the case of an atom consisting of a nucleus of mass M and an electron of mass m the reduced mass, $Mm/(M+m)$, has to be taken into account. It is easily understandable that this quantity will differ slightly, if we take an isotope of mass M^1 instead of M, and Urey, Brickwedde and Murphy (1932) found a displacement just such as would be predicted by this effect in the Balmer lines of the hydrogen isotope H^2 from those of the abundant isotope H^1.

In the case of band spectra the situation is very different. As there are two nuclei in the diatomic molecule, and as the moment of inertia differs largely with different isotopes and directly affects the vibration-rotation spectrum, we should expect Jevons's first factor to have a large effect. (There will, of course, also be an electronic isotope effect analogous to the electronic isotope effect of line spectra. As this is common to the lines of the same band system and will subtract out in the process of taking term-differences, it is not considered here.)

There is no *a priori* reason why the second factor should produce a detectable effect in hydrogen. The coupling between spin and orbits is very weak for light atoms, and should be weaker still for light molecules for which the distance between nuclei and excited electrons is generally greater than for the corresponding atoms (see pp. 26 and 47 of *Molecular*

Hydrogen and its Spectrum). Although the existence of nuclear spin has a profound effect on band structure and on the intensity sequence of band lines, it has very little effect on energy levels, showing itself, as far as they are concerned, in the fine structure of band lines. It is only in recent years that Richardson and Williams (1931) have shown that the lines of the triplet spectrum of hydrogen have detectable fine structure; it may be many years before an isotopic effect on fine structure can be detected experimentally.

The third factor, viz. slight difference of extranuclear electric field, is one about which we know less, and which is difficult to deal with, because it does not lend itself to any simple mathematical treatment. It is reasonable to suppose that this factor will be more important for the isotopes of hydrogen than for those of any other molecule. If we imagine a complicated nucleus consisting of a large number of protons and neutrons associated together and giving an extranuclear electric field measurable by the number of protons, we should not expect the addition of one neutron to such a nucleus to alter the field very much. It does not, however, follow that the field of the deuteron is likely to resemble closely the field of a single proton. If, as is now generally supposed, the deuteron consists of one proton and one neutron, from the fact that there are two bodies we should expect to find at least one direction of symmetry in the field of the deuteron. The field of the deuteron need not therefore closely resemble the isotropic field of the proton.

When it is found that the configuration constants of H_2 and D_2 differ so widely, we are forced to the conclusion that this result must be explained by the difference of extranuclear field. As, however, in the process of calculating the configuration constants, we have represented the spectroscopic term by an expansion in powers of the quantum numbers (equation (3) of Part I):

$$F(v, K) = \sum_{ij} Y_{ij}(v+\tfrac{1}{2})^i K^j (K+1)^j, \quad . \quad . \quad . \quad (2)$$

and so have to deal with a large number of coefficients Y_{ij}, there remains the possibility that our method of choosing the Y_{ij} may have been at fault, or, to put the matter differently, there is a possibility that with a different set of Y_{ij} we could have obtained an equally good fit and have found the configuration constants to come out the same for the two isotopes.

Fortunately we are able to put the last-mentioned hypothesis to an experimental test. In order to do this we proceed on the usual assumption that the mass effect is the only isotope effect which need be taken into

account, and try to find out whether we can represent the three isotopes, H_2, HD, and D_2, by the same potential function.

If we take m to be the mass of the proton, and $2m$ to be that of the deuteron, the reduced masses of the H_2, HD, and D_2 molecules are

$$\begin{aligned} \mu_{H_2} &= m/2, \\ \mu_{HD} &= 2m/3, \\ \mu_{D_2} &= m. \end{aligned} \qquad (3)$$

It is true that these expressions do not take into account the fact that the mass of the deuteron is not $2m$ but $1{\cdot}99902m$ (see equation (14) of Part I). Nor do they take into account the masses of the external electrons. If we take the known mass of the deuteron into account, we have to write the reduced mass of the D_2 molecule as $m \times 1{\cdot}0002719$. Similarly we have to write the reduced mass of the HD molecule as $(2m/3) \times 1{\cdot}0000906$. These corrections would, however, make only a small difference to the results. As we cannot in any case make any accurate assessment of the effect of the external electrons on the reduced masses of the isotopic molecules, the simple expressions (3) have been retained.

To convert any one of the band constants of the H_2 molecule to the corresponding one of the HD molecule we must multiply it by the appropriate power of the mass factor:

$$\begin{aligned} \rho_1 &= \sqrt{\mu_{H_2}/\mu_{HD}}, \\ &= \sqrt{3}/2, \\ &= 0{\cdot}8660254, \end{aligned} \qquad (4)$$

and similarly, to convert from H_2 to D_2, by the appropriate power of the mass factor:

$$\begin{aligned} \rho_2 &= \sqrt{\mu_{H_2}/\mu_{D_2}}, \\ &= 1/\sqrt{2}, \\ &= 0{\cdot}7071068. \end{aligned} \qquad (5)$$

To understand how the idea of mass factor comes in we have merely to note that we are assuming the potential function of equation (1) to be common to the three isotopes, while the fundamental constant B_e is not so. As this constant is equal to $h/8\pi^2\mu r_e^2 c$, it contains the reduced mass of the molecule in the denominator. We should therefore have

$$\frac{B_e \text{ for HD}}{B_e \text{ for } H_2} = \frac{\mu_{H_2}}{\mu_{HD}} = \rho_1^2. \qquad (6)$$

It is a necessary part of the whole assumption that we must regard the internuclear distance r_e as being the same for the three isotopes, since

we should not expect the potential function to be the same, unless this were so.

The constants of analysis ω_e and u_e both involve B_e, since $\omega_e = 2\sqrt{a_0 B_e}$ and $u_e = 2B_e/\omega_e = \sqrt{B_e/a_0}$. With the help of these relations we can easily, from an inspection of Dunham's theoretical expressions for the Y_{lj}, build up a table showing the correct power of the mass factor by which each Y_{lj} of H_2 must be multiplied to give the corresponding Y_{lj} for the particular isotope in question. Table I gives this information for the major terms of the Y_{lj}. For the correction terms we must increase the power of ρ by 2.

In what follows we shall consider the $1s\sigma 2s\sigma\, {}^3\Sigma$ state of the three isotopes. For this state we have found in Part I that the a_0 constants of H_2 and D_2 differ very little, this state accordingly seems to provide the best material for the test to be applied. The observed term-differences for HD and D_2 given in this paper are extracted from the measurements of Dieke and Blue (1935).

TABLE I.—POWER OF THE MASS FACTOR BY WHICH THE MAJOR PART OF EACH Y_{lj} MUST BE MULTIPLIED TO GIVE THE CORRESPONDING Y_{lj} FOR THE ISOTOPES.

	$j=0$.	1.	2.	3.
$l=0$		ρ^2	ρ^4	ρ^6
1	ρ	ρ^3	ρ^5	
2	ρ^2	ρ^4	ρ^6	
3	ρ^3	ρ^5		
4	ρ^4	ρ^6		
5	ρ^5			
6	ρ^6			

The calculation of the Y_{lj} which will give the best fit for the three isotopes is lengthy but not difficult. Table II gives the Y_{lj} which were found to give the best fit. It was necessary to confine attention to the term-differences of low quantum number, since it was found useless to attempt to obtain a common formula which would give anything like a fit for the three isotopes, if the term-differences corresponding to higher values of the quantum numbers were taken into account.

Table II also gives the Y_{lj} for HD and D_2, which are simply the corresponding Y_{lj} for H_2 multiplied by the appropriate powers of the mass factors. Each correction term is given separately, because the power of the mass factor is different.

TABLE II.—Y_{lj} FOR THE $1s\sigma 2s\sigma\,{}^1\Sigma$ STATES OF H_2, HD, AND D_2 CALCULATED ON THE ASSUMPTION THAT THE Y_{lj} FOR HD AND D_2 ARE EQUAL TO THOSE OF H_2 MULTIPLIED BY THE APPROPRIATE POWERS OF THE MASS FACTORS.

						Correction Terms.	
		j=0.	1.	2.	3.	0.	1.
H_2	l=0		34·2752	−0·022565	0·0000197		−0·0526
	1	2669·20	−1·70079	0·0008193		−4·76	0·00072
	2	−72·1908	0·037377	*		0·7380	
	3	0·97118	−0·0024547			−0·08416	
	4	− 0·037695	0·0000115				
	5	0·0008925					
	6	− 0·0000181					
HD	l=0		25·7064	−0·012693	0·0000083		−0·0296
	1	2311·59	−1·10470	0·0003991		−3·09	0·00035
	2	−54·1431	0·021025	*		0·4151	
	3	0·63080	−0·0011958			−0·04100	
	4	− 0·021203	0·0000049				
	5	0·0004348					
	6	− 0·0000076					
D_2	l=0		17·1376	−0·005641	0·0000025		−0·0131
	1	1887·41	−0·60132	0·0001448		−1·68	0·00013
	2	−36·0964	0·009344	*		0·1845	
	3	0·34335	−0·0004339			−0·01488	
	4	− 0·009424	0·0000017				
	5	0·0001578					
	6	− 0·0000023					

* Too small to affect results.

Table III shows how far the Y_{lj} of Table II fit the observed rotational term-differences. The top figure is in each case the observed term-difference, while the one below it is the residual obtained by subtracting the calculated value from the observed term-difference.

The rotational term-differences are given first, because the reciprocal-action Y_{lj} were determined from them for the reason that this procedure is a little easier to carry out numerically than the alternative procedure of determining the reciprocal-action coefficients from the vibrational term-differences. The rotational term-differences for $\Delta K = 6 - 4$ were not taken into account in the calculation. Those for which $v = 1$ and 2 are given, as the experimental values are known for all three isotopes. In spite of not having been used in the calculation these term-differences show a moderately good fit.

Although the fit in Table III at first sight seems good, it is not really so. The residuals for H_2 and D_2 tend to be positive, and those for HD to be negative. There is also a drift in the signs of the residuals for H_2 from positive to negative in passing from the left-hand side of the table to the right-hand side. In the case of HD the drift is from negative

to more negative in a direction roughly from the right-hand top corner to the left-hand bottom corner. There is also a drift in the signs of the

TABLE III.—ROTATIONAL TERM-DIFFERENCES FOR THE $1s\sigma 2s\sigma\,^3\Sigma$ STATE.

	ΔK.	v=0.	1.	2.	3.	4.
H_2	2 − 0	199·51 0·01	189·75 0·02	180·26 −0·01	171·05 −0·01	162·0 0·0
	3 − 1	330·76 0·01	314·54 0·01	298·83 −0·01	283·57 0·03	268·5 0·0
	4 − 2	459·44 0·01	436·91 0·04	415·05 0·01	393·73 −0·03	372·5 −0·4
	5 − 3	584·64 0·02	555·85 0·00	528·02 0·00	500·90 0·01	
	6 − 4		670·91 0·25	636·93 −0·07		
HD	2 − 0	150·29 −0·04	143·87 −0·08	137·76 0·01	131·70 −0·01	125·76 −0·01
	3 − 1	249·63 0·07	238·94 −0·02	228·66 −0·01	218·62 0·00	208·74 −0·01
	4 − 2	347·29 −0·05	332·53 −0·03	318·21 0·00	304·24 0·03	290·43 −0·03
	5 − 3	443·07 −0·03	424·16 −0·06	405·83 −0·06	387·98 −0·03	
	6 − 4		513·42 −0·04	491·17 −0·07		
D_2	2 − 0	100·74 −0·02	97·28 0·02	93·89 0·04	90·54 0·02	87·24 0·00
	3 − 1	167·51 0·02	161·63 −0·04	156·01 0·01	150·46 0·01	144·99 −0·01
	4 − 2	233·57 0·01	225·48 0·04	217·54 0·02	209·77 −0·01	202·12 −0·06
	5 − 3	298·76 0·03	288·34 0·02	278·22 0·03	268·26 −0·02	
	6 − 4		350·16 0·06	337·80 0·03		

residuals for D_2, since a group of negative residuals occurs in the right-hand bottom corner.

Such drifts are evidence that the higher Y_{ij} have not been estimated

correctly. The fact that the drift is different for the different isotopes indicates that we are attempting to do the impossible in forcing the same potential function on the three.

This conclusion is borne out by consideration of the vibrational term-differences, from which the pure-vibration coefficients Y_{l0} were determined after substituting the reciprocal-action Y_{lj} found from the rotational term-differences. The vibrational term-differences are given in Table IV. Those for $\Delta v = 5-4$ were not used in the calculation and show large residuals. The values for $K=1$, 2, and 3, which are known for all three isotopes, are given.

In Table IV we see that the rows are consistent, but there are deviations in the residuals from row to row. This is inevitable from the manner in which the Y_{lj} have been fitted. After substituting the reciprocal-action terms in the expressions for the members of each row we are left with a function of the pure-vibration coefficients Y_{l0}, which should be the same for every member of the same horizontal row. Any variation in the sign or magnitude of the residuals along any row (apart from random variation due to observational errors) is therefore simply evidence that the reciprocal-action terms do not fit properly. The variations from row to row, however, mean that pure-vibration coefficients Y_{l0} cannot be found which will obey the mass-effect law and simultaneously fit all three isotopes.

The residuals for D_2 are positive and those for HD are negative, while the residuals for H_2 drift from positive to negative in passing down the table from row to row. Moreover, there is an alternation $+-+-$ in the successive signs of the first four rows of the H_2 table, which suggests that there is something peculiar about this state—perhaps a slight perturbation of the vibrational level $v=2$, but the evidence for this is not at all conclusive.

Tables III and IV establish the fact that the mass-effect law is quite sufficient to give a first rough analysis, provided that we confine attention to term-differences corresponding to low values of the quantum numbers. If the term-differences corresponding to higher values of the quantum numbers are taken into account, the mass-effect law is inadequate. The term-differences corresponding to low values of the quantum numbers are exactly those in which the higher Y_{lj} play an unimportant part. With increasing values of the quantum numbers the higher Y_{lj} become increasingly important. We therefore conclude that the Y_{lj}, for which l and j are small, fit the three isotopes moderately well, whereas the higher Y_{lj} do not.

By referring to the theoretical expressions for the Y_{lj} given by Dunham (1932) we see that the lowest Y_{lj} are those in which the constants,

$a_1, a_2, a_3, \ldots$, of the potential function play an unimportant part, whereas, if we increase l or j—particularly l—these constants have more and more

TABLE IV.—VIBRATIONAL TERM-DIFFERENCES FOR THE $1s\sigma 2s\sigma\,{}^3\Sigma$ STATE.

	Δv.	K=0.	1.	2.	3.	4.
H_2	1 − 0	2524·32	2521·07	2514·55	2504·85	2492·00
		0·08	0·10	0·08	0·09	0·10
	2 − 1	2388·26	2385·12	2378·80	2369·41	2357·06
		−0·03	−0·01	−0·04	−0·04	0·05
	3 − 2	2256·10	2253·05	2246·91	2237·77	2225·60
		0·04	0·07	0·07	0·09	0·04
	4 − 3	2126·86	2123·85	2117·78	2108·83	2096·57
		−0·02	0·00	−0·03	0·03	−0·30
	5 − 4		1996·57	1990·58	1981·75	
			−0·44	−0·44	−0·32	
HD	1 − 0	2202·82	2200·65	2196·37	2190·01	2181·63
		−0·04	−0·08	−0·11	−0·11	−0·06
	2 − 1	2100·04	2097·95	2093·84	2087·65	2079·46
		−0·09	−0·11	−0·10	−0·12	−0·13
	3 − 2	1999·97	1997·95	1993·93	1987·91	1979·91
		−0·05	−0·06	−0·05	−0·05	−0·07
	4 − 3	1902·09	1900·08	1896·15	1890·25	1782·39
		−0·06	−0·08	−0·06	−0·04	−0·06
	5 − 4		1803·83	1799·97	1794·13	
			−0·34	−0·29	−0·28	
D_2	1 − 0	1814·90	1813·80	1811·44	1807·97	1803·29
		−0·03	0·04	0·01	0·03	−0·02
	2 − 1	1745·80	1744·67	1742·43	1739·03	1734·50
		0·00	0·00	0·04	0·03	0·02
	3 − 2	1678·26	1677·18	1674·95	1671·63	1667·20
		0·03	0·06	0·05	0·06	0·04
	4 − 3	1612·10	1610·99	1608·81	1605·58	1601·19
		0·06	0·05	0·05	0·09	0·04
	5 − 4		1545·95	1543·73	1540·51	
			−0·01	−0·08	−0·07	

effect on the value of the coefficient. Y_{10} and Y_{01} do not involve the a's at all except in their correction terms. Y_{11} involves only a_1 in its major term, Y_{20} only a_1 and a_2, and so on.

There are two corrections to be made to our treatment of the mass effect, since (1) the mass of the deuteron has been taken approximately as twice that of the proton, and (2) the masses of the external electrons have been neglected. These corrections are small, and the fact that they have not been made should produce a slight, but systematic, source of misfit. The actual fit found is too irregular to be rectified by small corrections to the reduced masses of the three isotopes. The conclusion is inevitable that the constants of the potential function differ for the three isotopes.

As has been shown in Part I, there seems to be a genuine, though small, difference in the a_0 constant of the potential function. However this may be, there is a very appreciable difference in the configuration constants, a_1, a_2, a_3, . . ., which can only arise from Jevons's third factor, viz. difference in the extranuclear field.

The writer believes that the influence of extranuclear field plays a larger part in the structure of the bands of molecular isotopes than is generally admitted. Unfortunately, the rotating-vibrator model does not provide a very satisfactory basis for the study of this effect, as from its very nature it takes the field for granted and represents it by a set of arbitrary constants. The most we can hope to do with the help of the model is to find these arbitrary constants as accurately as possible, and search for empirical relations between them. It is intended to deal with this question in a subsequent paper.

Summary.

An analysis of the $1s\sigma 2s\sigma\, {}^3\Sigma$ state of the three isotopes, H_2, HD, and D_2 (the last two from the measurements of Dieke and Blue (1935)) has been carried out by the method of Dunham (1932), and it has been found: (1) that it is possible with an application of the ordinary mass-effect theory of molecular isotopes to represent the lowest-quantum term-differences of the three isotopic states with fair accuracy; and (2) that it is not possible so to represent the higher-quantum term-differences.

Conclusion (1) implies that the internuclear distance in the three isotopic molecules cannot differ very much. Conclusion (2) implies that the potential functions of the three isotopes are different, and means in fact that the field of the deuteron differs appreciably from that of the proton.

This paper is taken from a thesis presented to the University of St Andrews in application for the degree of D.Sc.

REFERENCES TO LITERATURE.

CRAWFORD, F. H., and JORGENSEN, Jr., T., 1936. "The Band Spectra of the Hydrides of Lithium," *Phys. Rev.*, vol. xlix, pp. 745–752.

DIEKE, G. H., and BLUE, R. W., 1935. "Fulcher Bands of HD and D_2," *Phys. Rev.*, vol. xlvii, p. 261.

DUNHAM, J. L., 1932. "The Energy Levels of a Rotating Vibrator," *Phys. Rev.*, vol. xli, p. 721.

JEVONS, W., 1932. *Report on Band Spectra of Diatomic Molecules*, Camb. Univ. Press.

RICHARDSON, O. W., 1934. *Molecular Hydrogen and its Spectrum*, Yale Univ. Press.

RICHARDSON, O. W., and WILLIAMS, W. E., 1931. "Fine Structure of the Hydrogen Band Lines," *Nature, London*, vol. cxxvii, p. 481.

SANDEMAN, I., 1938. "The Molecular Spectra of the Hydrogen Isotopes: I—Application of the Rotating-vibrator Model to the States of D_2," *Proc. Roy. Soc. Edin.*, vol. lix, pp. 1–14.

UREY, H. C., BRICKWEDDE, F. G., and MURPHY, G. M., 1932. "Hydrogen Isotope of Mass 2 and its Concentration," *Phys. Rev.*, vol. xxxix, p. 164.

(*Issued separately May* 20, 1939.)

XIII.—Expressions for Generalised Hypergeometric Functions in Multiple Series. By Professor T. M. MacRobert, D.Sc.

(MS. received December 30, 1938. Revised MS. received March 20, 1939. Read May 1, 1939.)

§ 1. *Introductory.*—In a recent paper (*Proc. Roy. Soc. Edin.*, vol. lix, 1939, pp. 49–54) expressions in multiple series with argument $1-z$ were found for Generalised Hypergeometric Functions of the type

$$ {}_{p+1}F_p(\alpha_r;\ \rho_s;\ z), $$

where $p \geq 1$. These formulæ are generalisations of known formulæ for the ordinary hypergeometric function $F(\alpha,\ \beta;\ \rho;\ z)$, and they are established by induction. The same method will be here employed to obtain generalisations of other known formulæ, these generalisations being in the form of multiple series.

In section 2 generalisations for the three alternative forms of the hypergeometric function are found. From the first of these a generalisation of Saalschütz's Theorem is derived in section 3. In section 4 formulæ, due to Gauss and Whipple, for well-poised series are generalised; and in the following section terminating cases of these are continued into a domain containing the unit point. Many known formulæ for terminating series with unit argument can be deduced from the results of this paper.

§ 2. *Generalisations of the Four Forms of the Hypergeometric Function.*—It is well known that

$$ \begin{aligned} F(\alpha,\ \beta;\ \rho;\ z) &= (1-z)^{\rho-\alpha-\beta}F(\rho-\alpha,\ \rho-\beta;\ \rho;\ z), \quad . \quad . \quad (1) \\ &= (1-z)^{-\alpha}F(\alpha,\ \rho-\beta;\ \rho;\ \zeta), \quad . \quad . \quad . \quad (2) \\ &= (1-z)^{-\beta}F(\rho-\alpha,\ \beta;\ \rho;\ \zeta), \quad . \quad . \quad . \quad (3) \end{aligned} $$

where $\zeta = z/(z-1)$, these being the four forms of the hypergeometric function. The generalisation of equation (1) is

$$ {}_{p+1}F_p(\alpha_r;\ \rho_s;\ z) = (1-z)^{\sigma_p - s_{p+1}} \times \prod_{r=1}^{p} \sum_{n_r=0}^{\infty} \frac{(\sigma_r - s_r + \nu_{r-1};\ n_r)(\rho_r - \alpha_{r+1};\ n_r)(\alpha_{r+2};\ \nu_r)}{n_r!\ (\rho_r;\ \nu_r)} z^{n_r}, \quad (4) $$

where $(k;\ n) \equiv k(k+1)(k+2)\ .\ .\ .\ (k+n-1)$, $s_r = \alpha_1 + \alpha_2 + \ .\ .\ .\ + \alpha_r$, $\sigma_r = \rho_1 + \rho_2 + \ .\ .\ .\ + \rho_r$, $\nu_r = n_1 + n_2 + \ .\ .\ .\ + n_r$, $r = 1, 2, 3, \ .\ .\ ., \nu_0 = 0$, and

the expression $(\alpha_{r+2};\ \nu_r)$ does not appear in the innermost summation. This formula reduces to (1) when $p=1$.

On substituting from (4) in the integral

$$\int_0^1 t^{\alpha_{p+2}-1}(1-t)^{\rho_{p+1}-\alpha_{p+2}-1}\,{}_{p+1}F_p(\alpha_r;\ \rho_s;\ zt)\,dt,$$

where $R(\alpha_{p+2}) > 0$, $R(\rho_{p+1}-\alpha_{p+2}) > 0$, and then evaluating the integral and applying formula (1), formula (4) with $p+1$ in place of p is obtained. Thus (4) is established by induction. The restrictions can then be removed by applying analytical continuation.

Note.—The necessity of imposing these restrictions and then removing them may be avoided by using the equivalent contour integral, taken round the path $(1+, 0+, 1-, 0-)$ in the ζ-plane. If $|z| < 1$ the contour can be chosen so that $|z\zeta| < 1$ throughout the contour, and the singularity $1/z$ is outside the contour. Then, proceeding step by step, we find that all the series involved are absolutely and uniformly convergent.

The generalisation of (2) is

$$ {}_{p+1}F_p(\alpha_r;\ \rho_s;\ z) = (1-z)^{-\alpha_1}\prod_{r=1}^{p-1}\sum_{n_r=0}^{\infty}\frac{(\alpha_{r+2};\ \nu_r)(\rho_r-\alpha_{r+1};\ n_r)}{n_r!\,(\rho_r;\ \nu_r)}\zeta^{n_r} \times \sum_{n_p=0}^{\infty}\frac{(\alpha_1;\ \nu_p)(\rho_p-\alpha_{p+1};\ n_p)}{n_p!\,(\rho_p;\ \nu_p)}\zeta^{n_p}, \qquad (5)$$

where $\zeta = z/(z-1)$. The proof is similar to that of (4).

§ 3. *A Generalisation of Saalschütz's Theorem.*—Saalschütz's Theorem,

$$ {}_3F_2\left(\begin{matrix}\alpha,\ \beta,\ -n;\ 1\\ \rho,\ 1+\alpha+\beta-\rho-n\end{matrix}\right) = \frac{(\rho-\alpha;\ n)(\rho-\beta;\ n)}{(\rho;\ n)(\rho-\alpha-\beta;\ n)}, \qquad (6)$$

is obtained by multiplying (1) by

$$(1-z)^{\alpha+\beta-\rho}$$

and equating the coefficients of z^n. Similarly, on multiplying (4) by

$$(1-z)^{s_{p+1}-\sigma_p}$$

and equating the coefficients of z^n it is found that

$$ {}_{p+2}F_{p+1}\left(\begin{matrix}\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_{p+1},\ -n;\ 1\\ \rho_1,\ \rho_2,\ \ldots,\ \rho_p,\ s_{p+1}-\sigma_p+1-n\end{matrix}\right) = \frac{(\sigma_p-s_p;\ n)(\rho_p-\alpha_{p+1};\ n)}{(\sigma_p-s_{p+1};\ n)(\rho_p;\ n)}$$

$$\times\prod_{r=1}^{p-1}\sum_{n_r=0}^{n-\nu_{r-1}}\frac{(\sigma_r-s_r+\nu_{r-1};\ n_r)(\rho_r-\alpha_{r+1};\ n_r)(\nu_{r-1}-n;\ n_r)\prod\limits_{s=r+2}^{p+1}(\alpha_s+\nu_{r-1};\ n_r)}{n_r!\,(\sigma_p-s_p+\nu_{r-1};\ n_r)(1-\rho_p+\alpha_{p+1}+\nu_{r-1}-n;\ n_r)\prod\limits_{t=r}^{p-1}(\rho_t+\nu_{r-1};\ n_r)}, \qquad (7)$$

where, of course, n is a positive integer.

§ 4. *A Generalisation of a Formula of Whipple's.*—The formula (Whipple, *Proc. Lond. Math. Soc.*, vol. xxvi, 1926, p. 267)

$$ {}_3F_2\left(\begin{matrix} \alpha,\ \beta,\ \gamma;\ z \\ \alpha-\beta+1,\ \alpha-\gamma+1 \end{matrix}\right)=(1-z)^{-\alpha}{}_3F_2\left(\begin{matrix} \frac{1}{2}\alpha,\ \frac{1}{2}+\frac{1}{2}\alpha,\ 1+\alpha-\beta-\gamma;\ \lambda \\ \alpha-\beta+1,\ \alpha-\gamma+1 \end{matrix}\right), \qquad (8) $$

where $\lambda=-4z/(1-z)^2$, is itself a generalisation of Gauss's Formula

$$ F\left(\begin{matrix} \alpha,\ \beta;\ z \\ \alpha-\beta+1 \end{matrix}\right)=(1-z)^{-\alpha}F\left(\begin{matrix} \frac{1}{2}\alpha,\ \frac{1}{2}+\frac{1}{2}\alpha-\beta;\ \lambda \\ \alpha-\beta+1 \end{matrix}\right). \qquad (9) $$

This can be seen by putting $\gamma=\frac{1}{2}+\frac{1}{2}\alpha$ in (8).

It has been shown in a previous paper (*Phil. Mag.*, ser. 7, vol. xxvi, 1938, pp. 87, 88) that formula (8) can be deduced from formula (9). The series on the right of (8) and (9) converge within the inner loop of the curve

$$ 16(x^2+y^2)=\{(x-1)^2+y^2\}^2. \qquad (10) $$

The corresponding general formula is

$$ {}_{p+1}F_p\left(\begin{matrix} a_1,\quad a_2,\ \ldots,\ a_{p+1};\ z \\ a_1-a_2+1,\ a_1-a_3+1,\ \ldots,\ a_1-a_{p+1}+1 \end{matrix}\right)=(1-z)^{-a_1} $$
$$ \times\prod_{r=1}^{p-2}\sum_{n_r=0}^{\infty}\frac{(\frac{1}{2}a_1+\nu_{r-1};\ n_r)(\frac{1}{2}+\frac{1}{2}a_1-a_{r+1};\ n_r)\prod_{s=r+2}^{p+1}(a_s+\nu_{r-1};\ n_r)}{n_r!\prod_{t=r+1}^{p+1}(a_1-a_t+1+\nu_{r-1};\ n_r)}\lambda^{n_r} $$
$$ \times{}_3F_2\left(\begin{matrix} \frac{1}{2}a_1+\nu_{p-2},\ \frac{1}{2}+\frac{1}{2}a_1+\nu_{p-2},\ a_1-a_p-a_{p+1}+1;\ \lambda \\ a_1-a_p+1+\nu_{p-2},\ a_1-a_{p+1}+1+\nu_{p-2} \end{matrix}\right) \qquad (11) $$

This can be established by induction, using the method by means of which (8) was derived from (9) in the paper referred to above. It should be noted that, at the point in the proof where use was made of formula (1), formula (4) with $p=2$ is employed instead.

§ 5. *Expression in Multiple Series for a well-poised Terminating Series.*—The formula (Thomae, *Math. Ann.*, vol. ii, 1870, pp. 427–444; see also *Phil. Mag.*, ser. 7, vol. xxv, 1938, pp. 848–851)

$$ \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\rho)\Gamma(\sigma)}{}_3F_2\left(\begin{matrix} \alpha,\ \beta,\ \gamma;\ -z \\ \rho,\ \sigma \end{matrix}\right) $$
$$ (=)\sum_{\alpha,\beta,\gamma}\frac{\Gamma(\beta-\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\rho-\alpha)\Gamma(\sigma-\alpha)}\Gamma(\alpha)z^{-\alpha}{}_3F_2\left(\begin{matrix} \alpha,\ \alpha-\rho+1,\ \alpha-\sigma+1;\ -\frac{1}{z} \\ \alpha-\beta+1,\ \alpha-\gamma+1 \end{matrix}\right), \qquad (12) $$

where $-\pi\leqq \text{amp. } z\leqq\pi$ gives the analytical continuation of the function on the left from the region $|z|<1$ into the region $|z|>1$. The symbol $(=)$ is employed to indicate equality in the sense that the expressions on the two sides represent the same function, each in its own domain.

On putting $a_{p+1} = -n$, where n is a positive integer, in (11), and applying (12) to the generalised hypergeometric function on the right of (11), we find that the third function on the right of (12), that is the function corresponding to γ in the summation, vanishes identically, since $\rho - \gamma$ becomes $a_{p+1} + \nu_{p-2}$, which is always zero or a negative integer. Also, when $z = 1$, the part of the formula given by the second function on the right of (12), that corresponding to β in the summation, has the value zero, as it contains a factor $1 - z$. Hence

$$
{}_{p+1}F_p\left(\begin{matrix} a_1, a_2, \ldots, a_p, -n; 1 \\ a_1 - a_2 + 1, \ldots, a_1 - a_p + 1, a_1 + n + 1 \end{matrix}\right) = \frac{(a_1 + 1;\ n)(\tfrac{1}{2}a_1 - a_p + 1;\ n)}{(a_1 - a_p + 1;\ n)(\tfrac{1}{2}a_1 + 1;\ n)}
$$

$$
\times \prod_{r=1}^{p-2} \sum_{n_r=0}^{n-\nu_{r-1}} \frac{(\tfrac{1}{2}a_1 + \nu_{r-1};\ n_r)(\tfrac{1}{2} + \tfrac{1}{2}a_1 - a_{r+1};\ n_r)(\nu_{r-1} - n;\ n_r) \prod_{s=r+2}^{p} (a_s + \nu_{r-1};\ n_r)}{n_r!\,(\tfrac{1}{2} + \tfrac{1}{2}a_1 + \nu_{r-1};\ n_r)(a_p - \tfrac{1}{2}a_1 + \nu_{r-1} - n;\ n_r) \prod_{t=r+1}^{p-1} (a_1 - a_t + 1 + \nu_{r-1};\ n_r)} \qquad (13)
$$

(*Issued separately May* 20, 1939.)

XIV.—Studies on Reproduction in the Albino Mouse. II. Contributions on the Maturation of the Sperm Cells. By **Hugo Merton** (Heidelberg). *Communicated by* Professor F. A. E. Crew, M.D., D.Sc. (With Two Text-figures.)

(MS. received December 15, 1938. Read March 6, 1939.)

The morphological characters of mammalian sperm cells taken from the *ductuli efferentes* differ only slightly from the sperm derived from the *vas deferens*. However, it is known that spermatozoa from the *caput epididymis*, when kept in physiological salt solution, quickly become immotile, whereas those from the *cauda epididymis* retain their motility for a long time (Moore, 1928). During the slow passage through the epididymis the spermatozoa undergo a physiological process of maturation, which is said to occur under the influence of the epithelium of the epididymis and to result in a lesser susceptibility on the part of the spermatozoa to extraneous influences (Braus and Redenz, 1924; Redenz, 1926; and Lanz, 1929). Other authors maintain that this maturation of the spermatozoon is not conditioned by environmental influences (Young, 1931). In any case the spermatozoa achieve full functional ability only after they have reached the *cauda epididymis* and the *vas deferens*. These are the spermatozoa which enter the female genital tract at copulation, and thus it follows that spermatozoa for artificial insemination in the mouse must be taken from the *vas deferens* and *cauda epididymis*.

Morphologically there exists a slight difference between the apparently mature spermatozoa in the testis and those in the epididymis. In the former there is a drop-like swelling situated immediately behind the head (first described by G. Retzius in various mammals), whereas in the latter the droplet is found at the posterior end of the middle piece. The droplet has thus migrated along the middle piece during the passage through the epididymis (Redenz, 1925; Belonoschkin, 1934). Some authors assume that this swelling is merely a remnant of spermatid protoplasm not yet shed and has no further significance, but Retzius and Redenz consider that it serves to nourish the spermatozoa, while Popa (1931) regards it as being of importance for sperm locomotion

and terms it "equilibrator." In recent reviews (Romeis, 1926; Stieve, 1930) the opinion is expressed that this droplet is not found in all spermatozoa and that its presence is an indication of incomplete maturation (Walton, 1933). According to Seliwanowa (1934), the destruction of the "lipoid" capsule increases the motility of the spermatozoa but reduces markedly the duration of their life; this is the only view concerning the significance of the protoplasmic droplet which approaches our own conclusions, derived from a study of its origin, occurrence, and physiological significance, which will be described below.

If semen from the *cauda epididymis* of a mouse is diluted with Ringer's solution and after several hours a drop of it is examined under the microscope, some immotile spermatozoa will be found as well as many active ones. In all the motile spermatozoa the kinoplasmic droplet, as I choose to call it, is attached to the posterior end of the middle piece, whereas in the dead ones it is no longer present. These spermatozoa have either lost the droplet after death, or they have died because they lost it. In order to decide which of these alternatives is correct, the following experiment was carried out. A sperm suspension such as described above was sucked up several times by a glass syringe with a narrow glass needle, and afterwards driven out vigorously. The microscopic preparations of a suspension so treated contain almost exclusively dead spermatozoa without droplets, the isolated droplets being scattered throughout the liquid. Only a few spermatozoa are motile, and some of these still possess the kinoplasmic droplet. Since the number of active spermatozoa is very small, it is very easy to observe some selected individuals for a period of time. After a few minutes it is seen that the spermatozoa without the droplet, which had been as active as the rest, very soon become quiescent, then lose their ability to swim, and finally remain immobile and dead. In a second series of experiments the concentration of the Ringer solution in which the spermatozoa were suspended was varied. As long as the drop remained connected with the spermatozoa by a kind of filament or else increased in volume by an enclosure of liquid, locomotion was maintained, but stopped as soon as the drop had become completely separated or had ruptured.

These experiments show without doubt that the spermatozoa from the *cauda epididymis* retain their activity only as long as they possess the droplet. Therefore the kinoplasmic droplet cannot be regarded as a structure superficially attached to the middle piece, but must be looked upon as part of the kinoplasmic sheath surrounding it (fig. 2, *e*). For this reason the spermatozoa in hypertonic solutions always bend at the point at which the kinoplasmic droplet is attached

(fig. 1). If the surface-cover of the spermatozoon is ruptured by artificially removing the droplet, death will ensue.

If, on the other hand, we examine spermatozoa which have been taken from the uterus of the mouse after mating or after artificial insemination, it will be seen that only a small percentage of the spermatozoa are still equipped with the kinoplasmic droplet. Redenz (1925), Roemmele (1927), and Lagerlöf (1934) have made a similar observation on ejaculated bull spermatozoa. Apparently the drop is not required for the maintenance of locomotion of spermatozoa after ejaculation. (Incidentally, even completely mature spermatozoa from the *vas deferens* may lose their kinoplasmic droplet after several hours in a physiological salt solution without damage to themselves.) There is now no longer any difference in the duration of life between these two types of spermatozoa, and Roemmele even found an increased activity in spermatozoa without the droplet in the bull ejaculate. This shows that after ejaculation the loss of the droplet is not deleterious for the spermatozoa, but, on the contrary, results in increased activity. The process of kinoplasmic maturation of the middle piece has now reached the stage at which the droplet has become superfluous for the life of the spermatozoa and can be dispensed with without damage.

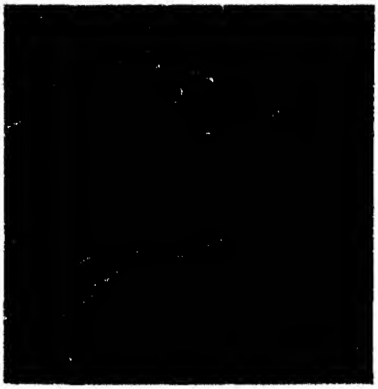


FIG. 1.—Spermatozoa from the *caput epididymis* in hypertonic Ringer solution. The middle piece is bent at the point of attachment of the kinoplasmic droplet.

In an investigation concerning the origin of the droplet and its importance, a study was made of the last stages of spermatid formation in the testis of the mouse. In the tubuli of 31-day-old males typical bundles of spermatids are anchored with their anterior end in the cytoplasm of the Sertoli cells. The rod-shaped head of the spermatid can be recognised by its stronger refraction of light; the head and the middle piece are enclosed by a common protoplasmic piriform mass, and to the free posterior end there is attached a delicate filament (fig. 2, *a*); the development of this can be observed, and it is seen occasionally to perform whip-like movements. In males a few days older the head has attained its typical shape (fig. 2, *b*). Soon afterwards the spermatid protoplasm, which has become superfluous, is expelled in several portions (fig. 2, *c*); it contains granular and globular inclusions. The spermatozoa, which

are still attached to the Sertoli cells, have now attained their final shape. If at this stage they are removed from their anchorage, a hemispherical droplet of homogenous appearance is seen to be attached to the neck-piece; in those which are found lying free in the tubuli the droplet has increased somewhat in size without changing its position. This is the kinoplasmic droplet described above. It is incorrect to regard it as a remnant of an expelled protoplasm for the following reasons: (1) Optically the droplet differs from the rest of the protoplasm of the spermatid

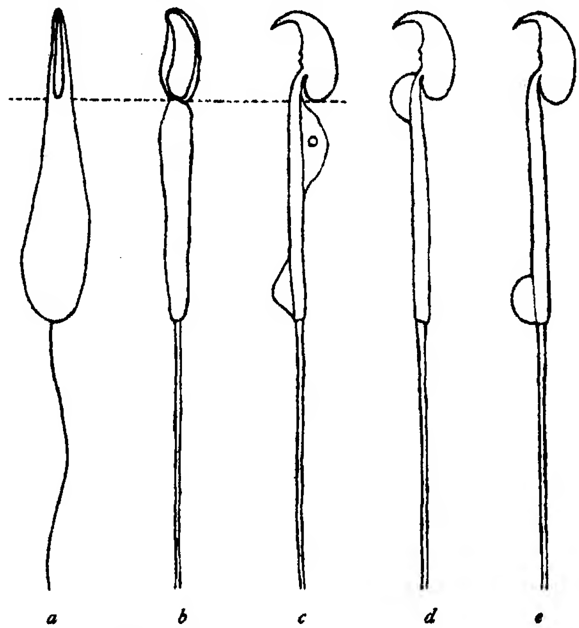


FIG. 2.—Last stages of the spermatid formation in the testis and epididymis of the mouse.

by its perfect homogeneity; (2) the droplet becomes visible only when the superfluous protoplasm has been expelled and the spermatozoon has attained its final shape; (3) it is not expelled like the rest of the protoplasm, but is maintained as an integral part of the spermatozoon as long as this remains in the male genital tract; (4) its premature loss leads to death of the spermatozoon.

Spermatozoa from the testis with proximal kinoplasmic swellings (fig. 2, *d*) lose their motility in Ringer's solution after a few minutes; the same holds true for similar spermatozoa from the upper section of the *caput epididymis*. As the spermatozoa approach the *vas deferens* their degree of motility and viability increase. Spermatozoa from the *vas deferens* and the *cauda epididymis* of the mouse (fig. 2, *e*) retain their motility in

Ringer at 18° C. for an average of 25 hours. The viability, however, varies considerably at lower temperatures, and in a medium which is rich in proteins and poor in electrolytes they may be kept alive for an even longer time.

Redenz (1925) and Lanz (1929) (*cf.* also Knaus, 1932) in particular maintained that spermatozoa from the testis and the rete testis in various mammals were endowed with sufficient activity to pass through the epididymis by virtue of their own energy. This assumption has been disputed, among others, by Moore (1928) and Young (1931); and is also contradicted by the fact that the ciliary current in the *ductuli efferentes* causes vortices and is not directed towards the testis (Zawisch-Ossenitz, 1932). Moreover, it should be remembered that the spermatozoa examined are always removed from the canal system of the male genital ducts, and their natural milieu is undoubtedly such that there is no question of an incitement to motility. Furthermore, as we have already seen, the motor centre of the spermatozoa in the testis is as yet hardly functional, and an energy consumption at this stage seems very improbable. The downward passage of the spermatozoa through the epididymis may occur through suction and pressure, and in the *cauda epididymis* contraction may be a factor (Belonoschkin, 1934). In this connection it may be recalled that it was shown only recently that the ciliated epithelia in the pharynx of the frog, a standard example of ciliary movement, "in the absence of extraneous factors retain an inactive quiescent state" (Lucas, 1933).

Discussion.

It appears that all mammalian spermatozoa possess the kinoplasmic swelling of the middle piece as long as they remain in the male genital tract. The presence of this kinoplasm is required for the functioning of the middle piece, which is generally considered as the motor centre of the spermatozoon. A spermatozoon taken from the male genital tract which loses its droplet, rapidly becomes immotile and dies, but after it has entered the female genital tract the loss of the droplet has no deleterious consequences. The kinoplasmic droplet is not a superfluous protoplasmic remnant of the spermatid, but an integral part of the spermatozoa in the male genital tract. In all the free spermatozoa from the testis the droplet is attached to the neck of the middle piece; during the transport of the spermatozoa through the epididymis it migrates along the intermediate piece but is not expelled at its posterior end. According to species, the migration of the swelling may be completed in the *caput* or only in the *cauda epididymis*, and it also varies according

to the frequency of ejaculation. According to Roemmele (1927), in the bull the second ejaculate, which followed rapidly after the first, contained 50 per cent. of spermatozoa with droplets, which means that these spermatozoa were not yet completely mature. Redenz (1925) has made similar observations on man, dog, and rabbit.

This process of maturation of the middle piece is not restricted to mammals. Merton (1926–27) has described corresponding processes for Selachians, Cephalopods, and Gastropods, and showed (1930, 1931) in great detail that in the pulmonate *Planorbis*, after expulsion of the protoplasm from the spermatids, the number of kinoplasmic bodies which leave the basal cell is equal to the number of spermatids attached to this cell. These kinoplasmic drops migrate along the long middle piece of the spermatids, and only after this process has been completed are the spermatozoa fully capable of functioning. This process is of particular interest, because it shows the importance of the auxiliary cells to which the spermatids are attached before they attain functional maturity. It is known that in mammals at the end of spermatogenesis all spermatids enter the Sertoli cells. These cells are said to be concerned, among other things, with the nutrition of the spermatids, but it appears incomprehensible why this particular stage should suddenly require special nutrition, and this view has been questioned by various writers (Korschelt, 1902; Bowen, 1922; Merton, 1926; Jaffé, 1932; and Moore, 1932). Moore (p. 305) says rightly: "The Sertoli cells have been termed 'nurse cells,' but practically nothing is known of the limitations of, or of the function of, the spermatozoon-Sertoli cell relationship." Many facts suggest that the Sertoli cells have the same function as the basal cells in the hermaphroditic gonad of the snail, namely, that of providing the middle piece of the spermatozoon with a protoplasm rich in lipoids, viz. the kinoplasm. Only with the aid of this substance can spermatozoa attain their full motility. This assumption is strengthened by the high lipoid content of the Sertoli cells. The crowding of the cell elements in the testicular tubes of mammals makes it impossible to demonstrate the validity of these views. In various other animal groups, however, the corresponding processes are seen so clearly that it seems that these generalizations are completely justified. It is submitted, therefore, that the function of the auxiliary cells in the male gonad is to provide the sperm cells with material for the production and functioning of a motor centre. The nutritive function is restricted to those auxiliary cells which are naturally equipped for it, viz. the nurse cells of the female gonads.

ACKNOWLEDGMENT.

These studies were carried out in 1937–38 in the Institute of Animal Genetics, University of Edinburgh. I have great pleasure in acknowledging the generous hospitality granted to me by Professor F. A. E. Crew in his Institute. My sincere thanks are also due the Society for the Protection of Science and Learning in London for the grant which enabled me to carry out the work.

SUMMARY.

1. All the spermatozoa from the *vas deferens* and the *cauda epididymis* show at the posterior end of the middle piece a protoplasmic swelling, "the kinoplasmic droplet."

2. Spermatozoa from the male genital tract which have lost the kinoplasmic droplet in physiological salt solution quickly die.

3. The kinoplasmic droplet is not a remnant of the expelled cellular protoplasm, but is a special substance which is required for the completion of the maturation process of the spermatozoa.

4. It is assumed that the Sertoli cells have no nutritive function, but that their main purpose is to provide the spermatozoon with material for the functioning of its motor centre, *i.e.* the middle piece.

REFERENCES TO LITERATURE.

BELONOSCHKIN, B., 1934. "Der gegenwärtige Stand der Spermatozoenforschung," *Arch. Gynäk.*, vol. clviii, pp. 181–214.

BOWEN, R. H., 1922. "Studies on Insect Spermatogenesis. II. The Components of the Spermatid, etc., in Hemiptera," *Journ. Morphol.*, vol. xxxvii.

BRAUS, H., und REDENZ, E., 1924. "Nebenhoden und Samenfäden," *Anat. Anz. Erg. H.*, vol. lviii, pp. 121–131.

JAFFÉ, R., und BERBERICH, F., 1932. "*Hoden*" *in Handb. d. inneren Sekretion v. Max Hirsch*, vol. i, p. 197.

KNAUS, H., 1932. "Zur Physiologie der Spermatozoen," *Arch. Gynäk.*, vol. cli, pp. 302–323.

KORSCHELT, E., und HEIDER, K., 1902. *Lehrbuch der vergl. Entwicklungsgesch.*, vol. i, pp. 486–487.

LAGERLÖF, N., 1934. *Morphologische Unters. üb. Veränderungen i. Spermatozoenbild des Hodens beim Bullen mit verminderter und aufgehobener Fertilität*, Upsala.

LANZ, TH. VON, 1929. "Die reelle Acidität in den einzelnen Abschnitten des männl. Genitalapparates d. Ratte u. ihre hormonale Bedingtheit," *Arch. ges. Physiol.*, vol. ccxxii, pp. 181–214.

LUCAS, A. M., 1933. "Comparison of Ciliary Activity under *in vitro* and *in vivo* Conditions," *Proc. Soc. Exp. Biol. Med.*, vol. xxx, pp. 501–506.

MERTON, H., 1926. "Die verschiedenartige Herkunft des Kinoplasmas der Samenzellen," *Biol. Zentrbl.*, vol. xlvi, pp. 656–678.

—— 1927. "Die Bewegungen der Samenzellen und ihr kinetisches Zentrum," *Zeits. Zellforsch. mikr. Anat.*, vol. v, pp. 629–664.

—— 1930. "Die Wanderung der Geschlechtszellen in der Zwitterdrüse von *Planorbis*," *ibid.*, vol. x, pp. 527–531.

—— 1931. "Untersuchungen üb. d. Entstehung und Natur des Kinoplasmas i. den Zwitterdrüsen der Pulmonaten," *ibid.*, vol. xiii, pp. 202–235.

MOORE, C. R., 1928. "On the Properties of the Gonads as Controllers of Somatic and Physical Characteristics. X. Spermatozoon Activity and the Testis Hormone," *Journ. Exp. Zool.*, vol. l.

—— 1932. "Biology of the testis." E. Allen: *Sex and Internal Secretions*, Baltimore.

POPA, GR. TTA., 1931. "A Contribution to the Biology of Spermatozoa," *Proc. 2nd Int. Congr. Sex Research.*

REDENZ, E., 1925. "Versuch einer biologischen Morphologie des Nebenhodens," *Arch. Entwmech.*, vol. cvi, pp. 290–302.

—— 1926. "Nebenhoden und Spermienbewegung," *Würzburg. Abh. N.S.*, vol. iv, pp. 108–149.

RETZIUS, G., 1909. "Biologische Untersuchungen an Spermien," *Biolog. Unters.*, 13 u. 14 Folge.

ROEMMELE, O., 1927. "Biologische und physiologische Untersuchungen am Sperma und am Scheidensekret des Rindes im Hinblick auf die künstliche Besamung," *Zool. Jahrb. Abt. allgem. Zool. Physiol.*, vol. xliv, pp. 85–148.

ROMEIS, B., 1926. "Hoden, samenableitende Organe und akzessorische Geschlechtsdrüsen," *Bethe, Handb. norm. path. Physiol.*, vol. xiv, p. 1.

SELIWANOWA, O. A., 1934. "Die Stabilität der Spermien landwirtschaftlicher Tiere," *Biol. Journ.*, vol. iii, pp. 630–637 (Russian with German summary).

STIEVE, H., 1930. "Der Hoden," *W. von Moellendorff. Handb. mikr. Anat.*, vol. vii, p. 1.

WALTON, A., 1933. "The Technique of Artificial Insemination," *Imp. Bur. Animal Genetics*, Edinburgh.

YOUNG, W. C., 1931. "A Study of the Function of the Epididymis. III. Functional Changes undergone by the Spermatozoa during their Passage through the *Epididymis* and *vas deferens* in the Guinea Pig," *Journ. Exp. Biol.*, vol. viii, pp. 151–162.

ZAWISCH-OSSENITZ, C., 1932. "Der Flimmerstrom in den ductuli efferentes des Hodens und die Bewegung der Spermien," *Zeits. mikr. anat. Forsch.*, vol. xxxii, pp. 84–106.

(*Issued separately May* 26, 1939.)

XV.—On Non-Associative Combinations. By I. M. H. Etherington, B.A.(Oxon.), Ph.D.(Edin.), Mathematical Institute, University of Edinburgh.

(MS. received October 18, 1938. Revised MS received March 24, 1939. Read May 1, 1939.)

CONTENTS.

§ 1. INTRODUCTION.

NUMEROUS combinatory problems arise in connection with a set of elements subject to a non-associative process of composition—let us say of multiplication—commutative or non-commutative.

Non-associative products may be classified according to their *shape*. By the shape of a product I mean the manner of association of its factors without regard to their identity. Shapes will be called *commutative* or *non-commutative* according to the type of multiplication under consideration. Thus if multiplication is non-commutative, the products (AB.C)D and (BA.C)D are distinct but have the same shape, while D(AB.C) has a different shape. The three expressions, however, have the same commutative shape. I confine attention to products (like these) in which the factors are combined only two at a time.

In § 2 I define addition and multiplication of shapes, and show that they may be regarded as the "positive integers" of a kind of non-associative arithmetic. With commutative multiplication this provides a convenient numerical notation by which shapes of great complexity can be easily specified.

A non-associative product or shape may be visualised as a *pedigree*, by which I mean a *tree* (Cayley, 1857) which (going from the root upwards, *i.e.* from the product to its factor elements) bifurcates at every knot (Cayley, 1859). Trees in general may quantifurcate arbitrarily at the knots, representing a more general kind of non-associative assemblage, which was also considered abstractly by Schröder (1870). A fourfold classification of shapes, arising partly out of this representation, is discussed in § 3.

Enumerative problems connected with non-associative combinations have been considered from various points of view by Catalan (1838, p. 515; etc.), Rodrigues (1838), Binet (1839), Schröder (1870): see Netto (1901, §§ 122–128) for a summary of their work; also by Cayley (1857, etc.), Wedderburn (1922). Some further enumerations are discussed here (§ 4); in particular, with the aid of the concept of ***mutability***, defined in § 3, it is shown that the commutative and non-commutative cases can be treated simultaneously. Thus equation (33) below, with y put equal to 1, yields the known formulæ (25), (27) for the commutative case; putting $y=2$, the known results (24), (26), (28) follow for the non-commutative case.

§ 2. Arithmetic of Shapes.

To eliminate brackets in writing non-associative products, it is convenient to use groups of dots to separate the factors when necessary, fewness of dots implying precedence in multiplication. Thus $\mathrm{A:.BC.AD^2:E}$ means $\mathrm{A\{[(BC)(A\overline{DD})]E\}}$. (The notation is due to Peano.)

Products and shapes in which the factors are absorbed one at a time (*e.g.* $\mathrm{A:BC.D:.E}$) will be called *primary*. The shapes generated by repeated squaring of an element, and products having such a shape (*e.g.* $\mathrm{AB.CD:EF.GH}$), will be called *plenary*. It will be seen in § 3 that all other shapes are in a sense intermediate between these two extremes.

For the moment, confine attention to the case of commutative multiplication, where a primary shape is unique when the number of factors δ is given. A power having this shape will be denoted X^δ: *e.g.* X^4 means $\mathrm{XX.X:X}$. All other powers can be represented by suitably partitioning the index, using brackets when necessary, with the following conventions: the product of two powers of the same element is indicated as a sum in the index, a power of a power as a product in the index, and an iterated power as a power in the index. Thus:

$$\begin{gathered}
X^{2+3}=X^2X^3,\\
X^{2\cdot 3}=(X^2)^3, \qquad X^{3\cdot 2}=(X^3)^2,\\
X^{2^3}=((X^2)^2)^2, \quad X^{3^2}=(X^3)^3,\\
X^{(2\cdot 2+3)+(2+3)^2}=X^2X^2.X^3::X^2X^3.X^2X^3:.X^2X^3.X^2X^3:X^2X^3.
\end{gathered}$$

Addition of indices, since it reflects non-associative multiplication of powers, is commutative but non-associative. On the other hand, multiplication of indices is non-commutative (as seen above), but associative, since $X^{ab.c}$ and $X^{a.bc}$ both mean $((X^a)^b)^c$, which can therefore be written X^{abc} unambiguously. This becomes X^{a^3} when $a=b=c$; and similarly with any number of factors in the index.

Further, $X^{a(b+c)}$ means $(X^a)^{b+c}$, *i.e.* $(X^a)^b(X^a)^c$, which is the same as X^{ab+ac}. Hence in the arithmetic of the indices

$$a(b+c)=ab+ac.$$

But in general

$$(b+c)a \neq ba+ca,$$

since $(X^bX^c)^a$ is not the same as $(X^b)^a(X^c)^a$. We may say therefore that in the arithmetic of the indices multiplication is predistributive with addition, but not in general postdistributive.

In these arguments a, b, c can be any expressions standing for complicated powers: they are not restricted to being simple integers indicating primary powers.

The notation provides an arithmetical method of specifying commutative shapes; for now the *shape* s of any commutative non-associative product can be redefined as the index of the corresponding power obtained by equating all the factors. The product $AB.C^2:D$, for instance, has the same shape as the power $(X^2)^2X=X^{2.2+1}$, namely $s=2.2+1$.

Consider what addition and multiplication of shapes mean when we are dealing with products in general instead of powers. Let Π_1, Π_2 be any two products with shapes s_1, s_2. Then s_1+s_2 is the shape of the product $\Pi_1\Pi_2$, while s_1s_2 is the shape of the product formed by substituting Π_1 for each of the factor elements of Π_2.

The procedure of this § may be described as a representation of the set of all commutative non-associative continued products formed from given elements on a non-associative arithmetic, whose integers are commutative shapes a, b, c, . . . with the rules of combination

$$\left.\begin{array}{lll} a+b=b+a, & ab.c=a.bc, & a(b+c)=ab+ac, \\ ab\neq ba, & (a+b)+c\neq a+(b+c), & (b+c)a\neq ba+ca. \end{array}\right\} \quad . \quad (1)$$

A similar representation is possible when multiplication of the original elements is non-commutative as well as non-associative. It is reflected as non-commutative addition of shapes, the other rules of combination (1) being unchanged. But the numerical specification of non-commutative shapes of increasing complexity rapidly becomes very complicated; to simplify it, some convention is required for distinguishing the $2^{\delta-2}$ distinct primary shapes of any given degree $\delta(>1)$.

§ 3. Classification of Shapes.

Shapes s will be classified by their *degree* $\delta(s)$, *altitude* $a(s)$, and *mutability* $\mu(s)$. Non-commutative shapes will be further classified by

the commutative shapes with which they are ***conformal***. These terms will now be defined.

The ***degree*** δ of a shape s means the number of factor elements in a product having this shape. It may be reckoned by evaluating s as if it were an integer in ordinary arithmetic.

Two non-commutative shapes s_1, s_2, which become the same shape s when multiplication is regarded as commutative, will be called ***conformal*** with each other and with s. Write $s_1 \sim s_2$ to indicate this. With commutative shapes, $s_1 \sim s_2$ means the same as $s_1 = s_2$, a commutative shape being conformal only with itself. The word is also applicable to products whose shapes are conformal. Thus

$$AB.C:D, \qquad A.BC:D, \qquad A:BC.D, \qquad A:B.CD$$

and their shapes

$$(2+1)+1, \qquad (1+2)+1, \qquad 1+(2+1), \qquad 1+(1+2)$$

are all conformal with the commutative power A^4 and its shape 4.

Let shapes be depicted as pedigrees (§ 1). Any non-associative product is then, so to speak, "descended from" its factors. The number of "generations" preceding the product itself is its ***altitude*** α (Cayley, 1875). At each knot in the pedigree two factors are united; the total number of knots is thus $\delta - 1$. Let a knot be called ***balanced*** if its two factors are conformal: then the number of unbalanced knots in a pedigree will be called its ***mutability*** μ. The various terms defined may be applied indiscriminately to the product, shape or pedigree.

If the mutability of any shape s (commutative or not) is μ, then there are evidently just 2^μ distinct non-commutative shapes which will become the same as s when multiplication is commutative. So μ could be defined alternatively as the logarithm to base 2 of the number of conformal non-commutative shapes.

If $s_1 \sim s_2$, then evidently

$$\delta(s_1) = \delta(s_2), \qquad \alpha(s_1) = \alpha(s_2), \qquad \mu(s_1) = \mu(s_2). \qquad (2)$$

The following formulæ are easily proved, r and s being any shapes, commutative or non-commutative, and ν an ordinary positive integer:—

$$\delta(r+s) = \delta(r) + \delta(s), \qquad (3)$$

$$\delta(rs) = \delta(r)\delta(s), \qquad (4)$$

$$\delta(s^\nu) = \delta(s)^\nu; \qquad (5)$$

$$\alpha(r+s) = 1 + \alpha(r) \quad \text{or} \quad 1 + \alpha(s) \quad \text{according as} \quad \alpha(r) > \text{ or } < \alpha(s), \qquad (6)$$

$$\alpha(rs) = \alpha(r) + \alpha(s), \qquad (7)$$

$$\alpha(s^\nu) = \nu\alpha(s); \qquad (8)$$

$$\mu(r+s) = 2\mu(s) \qquad \text{if } r \sim s, \qquad (9)$$

$$= 1 + \mu(r) + \mu(s) \text{ if not}, \qquad (10)$$

$$\mu(rs) = \delta(s)\mu(r) + \mu(s), \qquad (11)$$

$$\mu(s^\nu) = (1 + \delta + \delta^2 + \ldots + \delta^{\nu-1})\mu(s), \qquad (12)$$

where

$$\delta = \delta(s).$$

The last result is proved by induction from the preceding one. It may also be written

$$\frac{\mu(s^\nu)}{\mu(s)} = \frac{\tau(s^\nu)}{\tau(s)}, \qquad (13)$$

where

$$\tau = \delta - 1.$$

The degree, altitude and mutability can now be readily calculated for any given shape specified numerically. The table below gives all commutative shapes for which $\alpha \leqslant 4$, $\delta \leqslant 6$.

TABLE OF COMMUTATIVE SHAPES.

α.	δ.	μ.	s.
0	1	0	1
1	2	0	2
2	3	1	3
2	4	0	2.2
3	4	2	4
3	5	1	2.2 + 1
3	5	2	3 + 2
3	6	1	2.3
3	6	2	3.2
3	7	2	2.2 + 3
3	8	0	2^3
4	5	3	5
4	6	2	(2.2 + 1) + 1
4	6	3	(3 + 2) + 1, 4 + 2
Etc.			

As the table suggests, we cannot construct a shape with α, δ, μ assigned arbitrarily. Certain relations must be satisfied, namely:

$$2^\alpha \geqslant \delta \geqslant \alpha + 1; \text{ i.e. } \delta - 1 \geqslant \alpha \geqslant \log_2 \delta. \qquad (14)$$

$$\delta \geqslant \mu + 2, \textit{ except when } \delta = 1. \qquad (15)$$

$$\mu \leqslant 3.2^{\alpha-3} - 1; \text{ i.e. } \alpha \geqslant 3 + \log_2 \frac{\mu+1}{3}, \textit{ except when } \alpha < 3. \qquad (16)$$

δ is expressible as the sum of $\mu + 1$ powers of 2, not all alike if $\mu > 0$. (17)

(14) and (15) are easily proved by consideration of pedigrees. At one extreme, the equality $\delta = \alpha + 1$ holds only when s is primary; and the

same is true of $\delta=\mu+2$. Similarly at the other extreme, $\delta=2^a$, $\mu=0$ occur when and only when s is plenary.

(17) is proved by induction from (3), (5), (9), (10); it being noted that when $\mu=0$, s is of the form 2^a (plenary); when $\mu=1$, $s=2^a+2^\beta$ $(a\neq\beta)$; and that two like powers of 2 can be combined if desired into a single power of 2.

To prove (16), let μ_a be the greatest possible mutability for a shape whose altitude a is given; it will be shown that for $a \geqslant 3$

$$\mu_a=3.2^{a-3}-1.$$

In view of (2) it will be sufficient to consider only commutative shapes. By inspection of the table of commutative shapes,

$$\mu_0=\mu_1=0, \qquad \mu_2=1, \qquad \mu_3=2.$$

Now (see (6)) any shape of altitude $a+1$ is necessarily the sum of two shapes, one of altitude a and one of altitude $\beta < a$. By (9), (10), μ_{a+1} must be expressible either as $2\mu_a$ or as $1+\mu_a+\mu_\beta$. Since $\mu_1=0$, $\mu_2=1$, it follows that

$$\mu_{a+1}>\mu_a \quad \text{for} \quad a>0,$$

so that μ_a increases monotonically with a.

Now let a be any altitude (e.g. $a=3$) for which there exist at least three distinct shapes s_1, s_2, s_3 with the maximum mutability μ_a. Then

$$\mu(s_1)=\mu(s_2)=\mu(s_3)=\mu_a>\mu(s),$$

where s is any shape of lower altitude. Hence for the altitude $a+1$ also there will exist at least three distinct shapes of maximum mutability; namely,

$$s_1+s_2, \qquad s_2+s_3, \qquad s_3+s_1,$$

with the mutability given by (10)

$$\mu_{a+1}=1+2\mu_a. \qquad (a \geqslant 3.)$$

It follows that

$$1+\mu_{a+1}=2(1+\mu_a).$$

But

$$1+\mu_3=3,$$

whence

$$1+\mu_a=3.2^{a-3},$$

or

$$\mu_a=3.2^{a-3}-1 \qquad \text{if} \quad a \geqslant 3.$$

This proves (16).

It will be seen that the equality in (16) is attained by N_a commutative shapes of altitude a, where

$$N_{a+1}=\tfrac{1}{2}N_a(N_a-1), \qquad N_3=4. \qquad \text{(16a)}$$

§ 4. ENUMERATION OF SHAPES.

Let a_δ, p_α be the numbers of possible shapes of given degree δ and of given altitude α respectively, when multiplication is non-commutative and non-associative; and let b_δ, q_α be the corresponding numbers when multiplication is commutative and non-associative. Evidently

$$a_1 = b_1 = p_0 = q_0 = 1.$$

Remembering (3) and (6), and considering the different ways in which shapes of given degree or altitude can be formed from those of lower degree or altitude, we obtain the formulæ:

$$a_\delta = a_1 a_{\delta-1} + a_2 a_{\delta-2} + a_3 a_{\delta-3} + \ldots + a_{\delta-1} a_1, \qquad (18)$$

$$\left.\begin{aligned} b_{2\delta-1} &= b_1 b_{2\delta-2} + b_2 b_{2\delta-3} + \ldots + b_{\delta-1} b_\delta, \\ b_{2\delta} &= b_1 b_{2\delta-1} + b_2 b_{2\delta-2} + \ldots + b_{\delta-1} b_{\delta+1} + \tfrac{1}{2} b_\delta (b_\delta + 1), \end{aligned}\right\} \qquad (19)$$

$$p_{\alpha+1} = 2p_\alpha (p_0 + p_1 + p_2 + \ldots + p_{\alpha-1}) + p_\alpha^2, \qquad (20)$$

$$q_{\alpha+1} = q_\alpha (q_0 + q_1 + q_2 + \ldots + q_{\alpha-1}) + \tfrac{1}{2} q_\alpha (q_\alpha + 1). \qquad (21)$$

For $\delta = 1, 2, 3, \ldots$ and $\alpha = 0, 1, 2, \ldots$ the sequences start:

$a_\delta = 1,\ 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430,\ 4862, \ldots$
$b_\delta = 1,\ 1,\ 1;\ 2,\ 3,\ 6,\ 11,\ 23,\ 46,\ 98, \ldots$
$p_\alpha = 1,\ 1,\ 3,\ 21,\ 651,\ 457653,\ 210065930571, \ldots$
$q_\alpha = 1,\ 1,\ 2,\ 7,\ 56,\ 2212,\ 2595782, \ldots$

Let

$$F(x) = a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_\delta x^\delta + \ldots \qquad (22)$$

and

$$f(x) = -1 + b_1 x + b_2 x^2 + \ldots + b_\delta x^\delta + \ldots \qquad (23)$$

The following results are known:—

$$F(x)^2 - F(x) + x = 0, \qquad (24)$$

$$f(x)^2 + f(x^2) + 2x = 0; \qquad (25)$$

$$F(x) = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - 4x}, \qquad (26)$$

$$f(x) = \lim_{n \to \infty} -\sqrt{-2x + \sqrt{-2x^2 + \sqrt{-2x^4 + \ldots + \sqrt{-2x^{2^n} + 1}}}}, \qquad (27)$$

where in (27) each $\sqrt{\ }$ covers all that follows it;

$$a_\delta = \frac{(2\delta - 2)!}{(\delta - 1)!\ \delta!} = \frac{1}{\delta} \cdot {}^{2\delta-2}C_{\delta-1}. \qquad (28)$$

Of those formulæ, (18), (28) were given by Catalan (1838). (Catalan pointed out that a_δ is the number of ways in which a convex polygon of $\delta + 1$ sides can be divided up into triangles by diagonals. (28), as a consequence of (18) with $a_1 = 1$, was first established from this point of view, and was known to other writers, apparently first to Euler. Several papers on this topic appear in the *Journ. de Math.*, 1838–39.) Binet (1839)

introduced the generating function (22), and deduced (24), (26), (28) from (18). The calculations were repeated by Cayley (1859) from the pedigree point of view; by Schröder (1870); also by Wedderburn (1922), who discussed as well the commutative case, obtaining (19), (25), (27), and made a special study of the functional equation (25) and its more general solutions. (*Cf.* Etherington, 1937.)

It will now be shown that by introducing mutability we can discuss the commutative and non-commutative cases simultaneously and obtain a more general functional equation (33) which includes the two equations (24), (25) as special cases.

Let $c_{\delta\mu}$ be the number of possible commutative shapes of given degree δ and mutability μ, so that the corresponding number of non-commutative shapes will be

$$n_{\delta\mu} = 2^{\mu} c_{\delta\mu}. \qquad (29)$$

Then $n_{\delta\mu}$, $c_{\delta\mu}$ are defined for all integer values of δ, μ with $\delta > 1$, $\mu > 0$. For all other values of δ and μ, let $n_{\delta\mu}$, $c_{\delta\mu}$ be defined as zero.

Consider with the aid of (3), (9), (10) the different ways in which a non-commutative shape s of degree δ and mutability μ can be formed. Excluding $\delta = 1$, $\mu = 0$, $s = 1$, s must be of the form $s_1 + s_2$ where, by (3),

$$\delta(s_1) + \delta(s_2) = \delta.$$

If (10) held in all cases, we should have

$$1 + \mu(s_1) + \mu(s_2) = \mu,$$

and consequently

$$n_{\delta\mu} = \sum_{i, j, l, m} n_{il} n_{jm} \qquad (i + j = \delta, \quad 1 + l + m = \mu).$$

Subtracting the cases to which (10) does not apply, and adding those to which (9) does, we get as the correct formula

$$\left.\begin{aligned} n_{\delta\mu} &= \sum_{i, j, l, m} n_{il} n_{jm} - 2^{\frac{1}{2}(\mu-1)} n_{\frac{1}{2}\delta, \frac{1}{2}(\mu-1)} + 2^{\frac{1}{2}\mu} n_{\frac{1}{2}\delta, \frac{1}{2}\mu} \\ \text{where} \quad & i + j = \delta, \qquad l + m = \mu - 1, \qquad \delta \neq 1. \\ \text{Also} \quad & n_{10} = 1. \end{aligned}\right\} \qquad (30)$$

Putting $n_{\delta\mu} = 2^{\mu} c_{\delta\mu}$, and removing the factor 2^{μ},

$$\left.\begin{aligned} c_{\delta\mu} &= \tfrac{1}{2}\Big(\sum_{i, j, l, m} c_{il} c_{jm} - c_{\frac{1}{2}\delta, \frac{1}{2}(\mu-1)}\Big) + c_{\frac{1}{2}\delta, \frac{1}{2}\mu} \\ \text{where} \quad & i + j = \delta, \qquad l + m = \mu - 1, \qquad \delta \neq 1. \\ \text{Also} \quad & c_{10} = 1. \end{aligned}\right\} \qquad (31)$$

Now let

$$f(x, y) = \sum_{\delta, \mu} c_{\delta\mu} x^{\delta} y^{\mu}. \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Substituting (31) in (32), we obtain the functional equation

$$f(x, y) = x + \tfrac{1}{2}y\{f(x, y)\}^2 + (1 - \tfrac{1}{2}y)f(x^2, y^2). \quad . \quad . \quad (33)$$

Now, from the definitions of a_δ, b_δ, $c_{\delta\mu}$, $n_{\delta\mu}$,

$$a_\delta = \sum_\mu n_{\delta\mu} = \sum_\mu 2^\mu c_{\delta\mu}, \qquad b_\delta = \sum_\mu c_{\delta\mu}.$$

Consequently, comparing (22), (23), (32),

$$F(x) = f(x, 2), \qquad 1 + f(x) = f(x, 1). \quad . \quad . \quad . \quad (34)$$

It is readily verified that on putting $y=2$ the equation (33) reduces to (24); and that on putting $y=1$ it reduces to (25), as it should.

If on the right of (33) we substitute the first approximation

$$f(x, y) = x + \ldots, \qquad f(x^2, y^2) = x^2 + \ldots,$$

we obtain the second approximation

$$f(x, y) = x + x^2 + \ldots$$

Similarly the third approximation is

$$\begin{aligned} f(x, y) &= x + \tfrac{1}{2}y(x^2 + 2x^3 + x^4 + \ldots) + (1 - \tfrac{1}{2}y)(x^2 + x^4 + \ldots) \\ &= x + x^2 + x^4 + x^3 y + \ldots; \end{aligned}$$

and the process may be repeated to any required extent.

Alternatively, we may proceed in either of the following ways. Write

$$f(x, y) = xf_1(y) + x^2 f_2(y) + \ldots + x^\delta f_\delta(y) + \ldots \quad . \quad . \quad (35)$$

or

$$f(x, y) = g_0(x) + yg_1(x) + y^2 g_2(x) + \ldots + y^\mu g_\mu(x) + \ldots; \quad . \quad (36)$$

substitute in the functional equation (33), and equate coefficients. We obtain

$$\left.\begin{aligned} &f_1 = f_2 = 1, \quad f_3 = y, \quad f_4 = 1 + y^2, \\ &f_5 = y + y^2 + y^3, \quad f_6 = y + 2y^2 + 2y^3 + y^4, \ldots, \\ &f_{2\delta-1} = y(f_1 f_{2\delta-2} + f_2 f_{2\delta-3} + \ldots + f_{\delta-1} f_\delta), \\ &f_{2\delta} = y(f_1 f_{2\delta-1} + f_2 f_{2\delta-2} + \ldots + f_{\delta-1} f_{\delta+1} + \tfrac{1}{2} f_\delta^2) + (1 - \tfrac{1}{2}y) f_\delta(y^2); \end{aligned}\right\} . \quad (37)$$

$$\left.\begin{aligned} &g_0 = x(1 - x)^{-1}, \quad g_1 = x^3(1 - x)^{-1}(1 - x^2)^{-1}, \\ &g_2 = x^4(1 + x + 2x^2)(1 - x)^{-1}(1 - x^2)^{-1}(1 - x^4)^{-1}, \\ &g_3 = x^5(1 + x + 3x^2)(1 - x)^{-2}(1 - x^2)^{-1}(1 - x^4)^{-1}, \ldots, \\ &g_{2\mu-1} = g_0 g_{2\mu-2} + g_1 g_{2\mu-3} + \ldots + g_{\mu-2} g_\mu + \tfrac{1}{2} g_{\mu-1}^2 - \tfrac{1}{2} g_{\mu-1}(x^2), \\ &g_{2\mu} = g_0 g_{2\mu-1} + g_1 g_{2\mu-2} + \ldots + g_{\mu-1} g_\mu + g_\mu(x^2). \end{aligned}\right\} . \quad (38)$$

It will be observed that

$$f_\delta(2) = a_\delta, \qquad f_\delta(1) = b_\delta. \quad . \quad . \quad . \quad . \quad . \quad (39)$$

The first of these two methods is perhaps the quickest way of calculating many terms of the expansion of $f(x, y)$. By means of the second, we could find explicit formulæ for $c_{80}, c_{81}, c_{82}, c_{83}, \ldots$

With regard to the convergence of the various generating series, it may be observed that (22), since it is the expansion of (26), is absolutely convergent if $|x| < \frac{1}{4}$. Since $b_8 \leqq a_8$, it follows that (23) also converges absolutely if $|x| < \frac{1}{4}$; and since $f(x, 2) = F(x)$, it follows that the double series (32) converges absolutely if $|x| < \frac{1}{4}$, $|y| \leqq 2$.

Summary.

Non-associative combinations are classified and enumerated with the aid of a representation involving non-associative arithmetic.

REFERENCES TO LITERATURE.

BINET, M. J., 1839. "Réflexions sur le problème de déterminer le nombre de manières dont une figure rectiligne peut être partagée en triangles au moyens de ses diagonales," *Journ. de Math.* (1), vol. iv, pp. 79–90.

CATALAN, E., 1838. "Note sur une équation aux différences finies," *Journn. de Math.*, (1) vol. iii, pp. 508–516. Also 1839, vol. iv, pp. 91–94, 95–99; 1841, vol. vi, p. 74.

CAYLEY, A., 1857, 1859. "On the theory of the analytical forms called trees," *Phil. Mag.*, vol. xiii, pp. 172–176; vol. xviii, pp. 374–378. Also 1875, *Rep. Brit. Assoc. Adv. Sci.*, pp. 257–305; 1881, *Amer. Journ. Math.*, vol. iv, pp. 266–268; 1889, *Quart. Journ. Pure App. Math.*, vol. xxiii, pp. 376–378. (*Collected Math. Papers*, vol. iii, no. 203; iv, 247; ix, 610; xi, 772; xiii, 895.)

ETHERINGTON, I. M. H., 1937. "Non-associate powers and a functional equation," *Math. Gaz.*, vol. xxi, pp. 36–39, 153.

NETTO, E., 1901. *Lehrbuch der Combinatorik*, Leipzig.

RODRIGUES, O., 1838. "Sur le nombre de manières d'effectuer un produit de *n* facteurs," *Journ. de Math.* (1), vol. iii, p. 549.

SCHRÖDER, E., 1870. "Vier combinatorische Probleme," *Zeits. Math.*, vol. xv, pp. 361–376.

WEDDERBURN, J. H. M., 1922. "The functional equation $g(x^2) = 2ax + [g(x)]^2$," *Ann. Math.* (2), vol. xxiv, pp. 121–140.

(*Issued separately June* 27, 1939.)

XVI.—Cytogenetical Analysis of the Chromosomes in the Pig.

By **Professor F. A. E. Crew,** M.D., D.Sc., and **P. C. Koller,** Ph.D., D.Sc., Institute of Animal Genetics, University of Edinburgh. (With One Plate and Eleven Text-figures.)

(MS. received February 1, 1939. Read March 6, 1939.)

Introduction.

Now that it is accepted that the phenomena observed in genetical experimentation are repercussions of events which have occurred previously in the chromosomes, it is the rule confidently to appeal, whenever possible, to the evidence derived from cytological studies for final explanations of genetical behaviour. This being so, it follows that in those instances in which, for any reason, experimental breeding work is difficult, studies of the chromosomes during the division cycles may be expected to yield significant information concerning the peculiarities of hereditary transmission that might be expected were such genetical investigation undertaken. Further, the larger mammals of economic importance are, under present conditions, unsatisfactory genetical material; they are expensive, and commonly the details of structure and function which they present for examination are not easily or precisely definable and seem to obey no simple rule of inheritance. For these reasons it seems desirable that thorough cytological studies of these forms should be undertaken, for at least these are not costly and they may be expected to provide explanations of the apparently complicated genetic behaviour which these forms exhibit.

Material and Methods.

Testicular material from 8 Large White boars was obtained from the city abattoir. The ages of these animals at the time of slaughter ranged from 8 months to 2 years. As far as could be ascertained they were not closely related.

Several fixative solutions were used, the most satisfactory being Minouchi's and Champy's, with or without acetic acid. Sections were 16–18 μ and were stained with gentian violet. The drawings were made with the aid of an Abbe camera lucida, Zeiss 1·3 apochr. oil immersion objective and 30 × comp. eyepiece.

CHROMOSOME MORPHOLOGY.

In the metaphase plate of dividing spermatogonial cells, the larger members of the diploid chromosome set lie at the periphery, whilst the smaller ones are congregated toward the centre (fig. 1, *a*, *b*, *c*, *d*, and Pl., A, B, C.) This arrangement greatly facilitates counting. The chromosome number was determined in 5 spermatogonial cells of each of the 8 individuals: it was invariably 38. This is the number given by

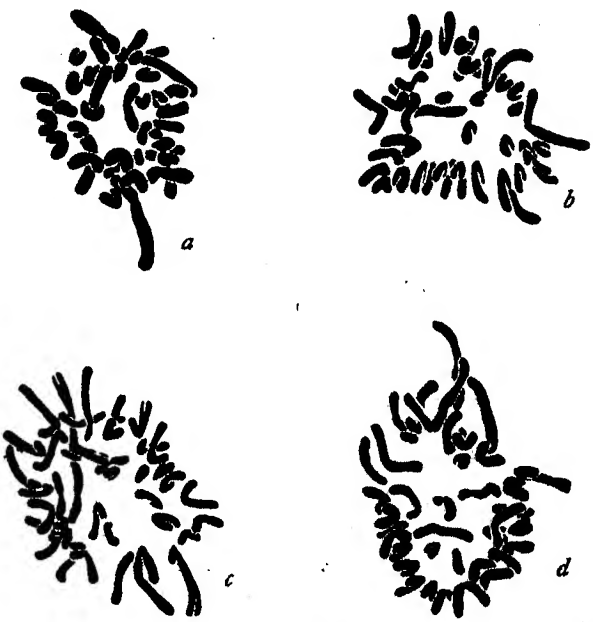


FIG. 1.—*a*, *b*, *c*, and *d*, mitotic metaphase plates of spermatogonial cells; *a* shows a metaphase plate in which the chromosomes are thicker and shorter than the chromosomes of the other plates. × 3500.

Krallinger (1931) and by Hillebrand (1936) and therefore is not in agreement with the observations of Wodsedalek (1913) and Hance (1917).

The longest chromosome (chromosome A) is 7–7·5 μ; B and C are 6 μ; D and E, 5·2–5·5 μ; F is 5 μ; five others are about 3–4 μ; and the shortest of them all is approximately 1·5–2 μ. One pair is to be distinguished from the rest for the reason that its members are of markedly different size. For this reason and also because of their peculiar behaviour during meiosis, these are regarded as the sex-chromosomes. The chromosomes differ among themselves also in respect of shape. It is established that the shape of a chromosome is largely determined by the position of the constriction that marks the point of the spindle attach-

ment (the centromere). There are 5 with median, 4 with submedian, and 9 with subterminal centromeres (fig. 2).

In the spermatogonial cells derived from one particular individual the chromosomes were shorter and stouter at metaphase than were those

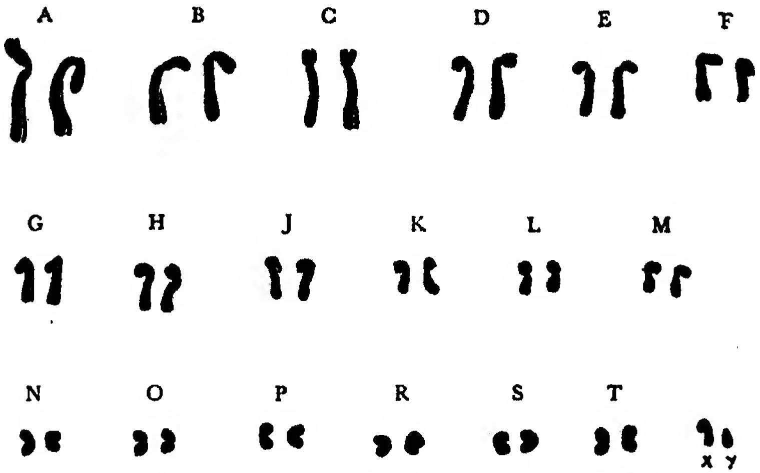


FIG. 2.—The diploid chromosome complement in the pig. The lengths of chromosomes A–F are given in the text. ×4000.

from any of the other boars (fig. 1, *a*). This greater contraction is regarded as a modification of the degree of spiralisation of the chromosome caused by gene action (Darlington, 1937).

CHIASMA FORMATION.

It is now generally accepted that chiasmata are formed during the meiotic prophase between associated homologous chromosomes and that they represent the points at which breakage and reunion of the paternal and maternal chromatids occur. Furthermore, normally chiasma formation is a *conditio sine qua non* of the orientation of the bivalents at the first meiotic metaphase. At the end of pachytene when chiasmata first appear in the bivalent as a result of repulsion between the two homologues, they actually indicate the points of exchange between partner chromatids. But during the ensuing diplotene, diakinesis and metaphase, the number of chiasmata commonly decreases and their positions change. This movement is known as the terminalisation of chiasmata (Darlington, 1937), since it is away from the centromere and towards the distal ends of the bivalent. In some forms, *e.g.* Campanula, all the chiasmata become terminalised; in others, *e.g.* Vicia, no such movement is observed; between these two extremes there are forms with

various grades of terminalisation, and the degrees can be determined by a comparison of the numbers of chiasmata present during pachytene, diplotene, diakinesis, and metaphase respectively. The terminalisation coefficient is obtained by dividing the numbers of terminal chiasmata by the total number of chiasmata, and is an index of the movement of

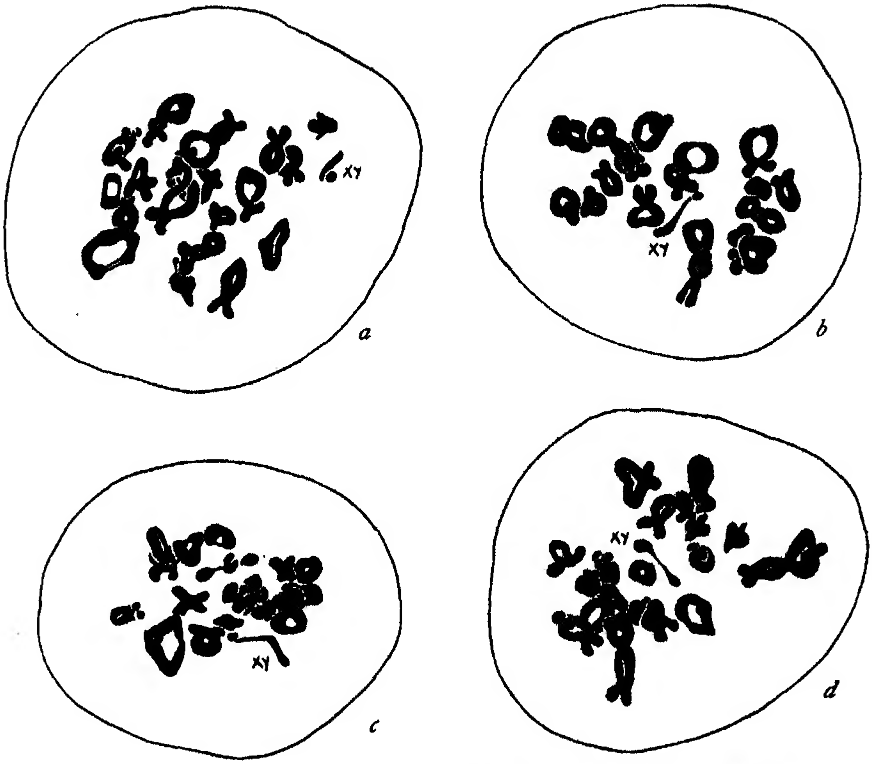


FIG. 3.—Mid-diplotene of meiosis in four primary spermatocytes. The sex bivalent is indicated. The autosomal bivalents show very similar structure in different cells. ×3500.

chiasmata and an indication of the initial chiasma frequency in the bivalents concerned.

Fig. 3, *a*, *b*, *c*, *d*, portray the mid-diplotene stage in the primary spermatocytes of the pig. Chiasma frequencies during mid-diplotene and metaphase were determined by counting the number of chiasmata in the autosomal bivalents of several primary spermatocytes (Table I).

The chiasma frequencies and the terminalisation coefficients of mid-diplotene and metaphase suggest that there is practically no movement of chiasmata during the two stages of meiosis. Such movement as occurs consists in a slight shift of the two distal chiasmata to the very

ends of a bivalent (fig. 4, *a*, *b*, *c*, and Pl., I, M), altering the shape of the bivalent without affecting the number of chiasmata.

TABLE I.—CHIASMA FREQUENCIES AT MID-DIPLOTENE AND METAPHASE.

Mid-diplotene.						Metaphase.					
Number of Bivalents with				Total No. Xta.	Total No. Term. Xta.	Number of Bivalents with				Total No. Xta.	Total No. Term. Xta.
4 Xta.*	3 Xta.	2 Xta.	1 Xa.			4 Xta.	3 Xta.	2 Xta.	1 Xa.		
..	3	15	..	39	24	1	2	15	..	40	27
1	2	15	..	40	26	..	2	16	..	38	25
1	2	15	..	40	25	..	2	15	1	37	26
1	2	14	1	39	25	1	2	15	..	40	29
..	3	15	..	39	23	..	3	15	..	39	28
3	12	74	1	197	123	2	11	76	1	194	135

Xa frequency per bivalent: 2·18.
term. coefficient: 0·62.

Xa frequency per bivalent: 2·15.
term. coefficient: 0·69.

* Xta = Chiasmata.

TABLE II.—CHIASMA FREQUENCIES IN FIVE INDIVIDUALS.

No. of Cells Analysed.	Age of Animal.	Xa Frequencies.		Term. Coefficient.	
		Mid-diplotene.	Metaphase.	Mid-diplotene.	Metaphase.
5	8 months	2·18	2·15	0·62	0·69
12	9 months	2·20	2·12	0·57	0·61
9	13 months	2·00	1·98	0·42	0·53
5	22 months	2·31	2·06	0·48	0·56
5	24 months	2·14	2·10	0·56	0·62
..	Average	2·16	2·06	0·53	0·61

Since the morphology of a bivalent is determined by the number and position of chiasmata in relation to the centromere, it follows that if terminalisation occurs this appearance will change during the successive phases of meiotic prophase. In the spermatocytes of the boar the

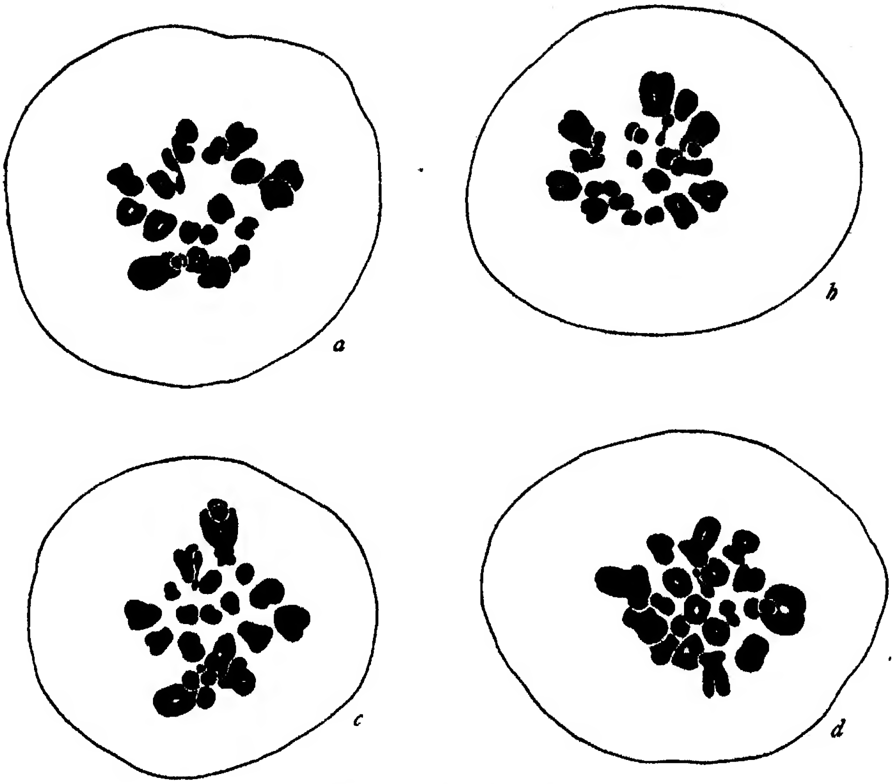


FIG. 4.—*a*, *b*, *c*, *d*, polar view of first metaphase of meiosis. The larger bivalents lie at the periphery of the spindle. × 3500.

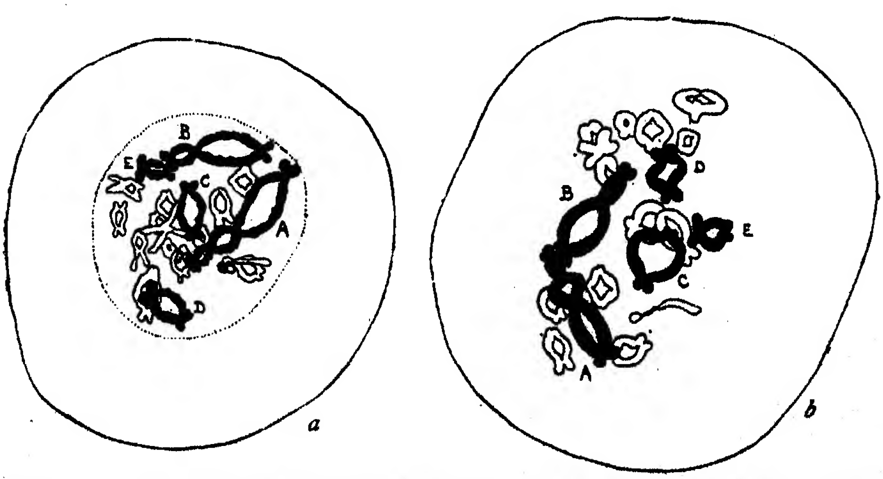


FIG. 5.—*a*, early diplotene stage, the five larger bivalents are solid black; *b*, late diplotene stage. The shape of the five bivalents is the same as it was during the early diplotene. × 3500.

shape of the bivalent at early diplotene is very similar to that at late diplotene, diakinesis, and metaphase (fig. 5, *a*, *b*).

A comparison of chiasma frequencies in five of the boars of different age and genetical constitution was then made (Table II).

There is nothing in these figures to suggest that the frequency is affected by age differences of this magnitude. Such slight variation as there is is likely to be a reflection of differences in genetical constitution or in environment. The relative uniformity of these figures supports the suggestion that the total number of chiasmata in the nuclei of the primary spermatocytes represents the total number of actual crossings-over.

It is possible that there may be a more or less direct proportionality between the mitotic length of the chromosome and the number of chiasmata. Thus the 2–3·5 μ chromosomes with median centromeres have 2 chiasmata, one in either arm: whereas the 7·5 μ one commonly has 4. A more exact analysis of this possible relationship could not be made because differences in length of the chromosomes are so gently graded.

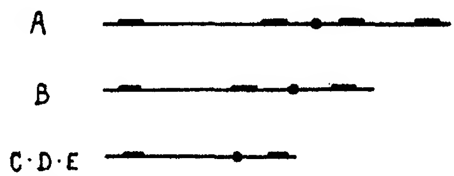


FIG. 6.—Diagram showing the approximate regions of the five larger bivalents where chiasmata usually are formed.

In order to determine whether the formation of chiasmata is restricted to definite regions of the bivalent, one and the same bivalent was studied in different nuclei. The bivalents, A, B, C, D, E (or F), the five largest were examined in 25 spermatocytes at mid-diplotene and metaphase. It was found that in different nuclei a given bivalent had the same number of chiasmata and also that the position of these chiasmata was apparently constant (fig. 5, *a*, *b*). This observation, admittedly incomplete, suggests that in the larger bivalents of the pig at least chiasmata are formed in restricted regions (fig. 6). Similar localisation of Xta is known in *Fritillaria*, *Mecosthetus*, in the sex-chromosomes of *Drosophila* and in many Tettigonidæ, but in these the localised regions are adjacent to the centromere, whereas in the boar they are localised in proximal and distal regions of the chromosome, as in Chrysochraom (Darlington, 1939).

Chromosome behaviour of this kind is necessarily followed by certain genetical repercussions. The degree of genetical linkage depends primarily on the spatial relationship of the genes concerned, and secondarily on the type and frequency of genetical crossing-over. Exceptionally high linkage between genes may be due either to irregular spacing of

the genes in the chromosome, *e.g.* Lebistes and Cepea, or to a restriction of exchange between partner chromatids due to chiasma localisation or by a structural change such as inversion. Only those genes lying at or near each side of the region of chiasma formation would show recombination and could therefore be mapped genetically. The other genes would exhibit complete linkage. Suppression or restriction of crossing-over alters the unit of inheritance. When there is free interchange between the paternal and maternal chromosomes the unit is the gene; when crossing-over is restricted the unit is the gene group (Darlington, 1937). Chiasma localisation limits recombination and this interferes with the creation of new character constellations, new types. This chiasma localisation may perhaps explain certain difficulties encountered in linkage studies with the pig.

The Sex Chromosomes.

The X, the longer member of the pair regarded as the sex-chromosomes, has a subterminal or submedian centromere. The Y, about two-thirds as long as the X, is rod-shaped and therefore probably has a

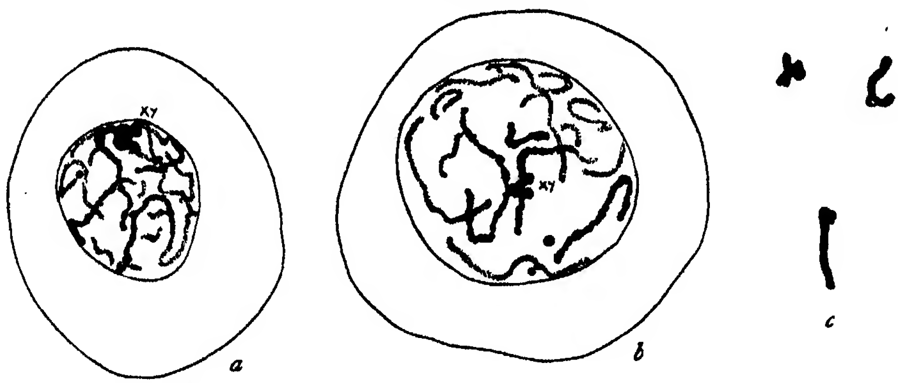


Fig. 7.—*a*, *b*, early and late pachytene, showing the sex bivalent; *c*, the configuration of sex bivalent at diplotene and diakinesis. ×3500.

terminal or nearly terminal centromere. Krallinger (1931) identified a large horse-shoe shaped chromosome in the spermatogonial metaphase as the X, but failed to identify the Y. The sex-chromosomes have a somewhat asymmetrical form during leptotene-pachytene and are associated with a small nucleolus (fig. 7, *a*, *b*, and Pl., E). This asymmetry becomes more pronounced at the end of pachytene and during diplotene, the X and Y being connected terminally (fig. 7, *c*, and Pl., L). They are stained just as deeply as are the autosomal bivalents. At metaphase the

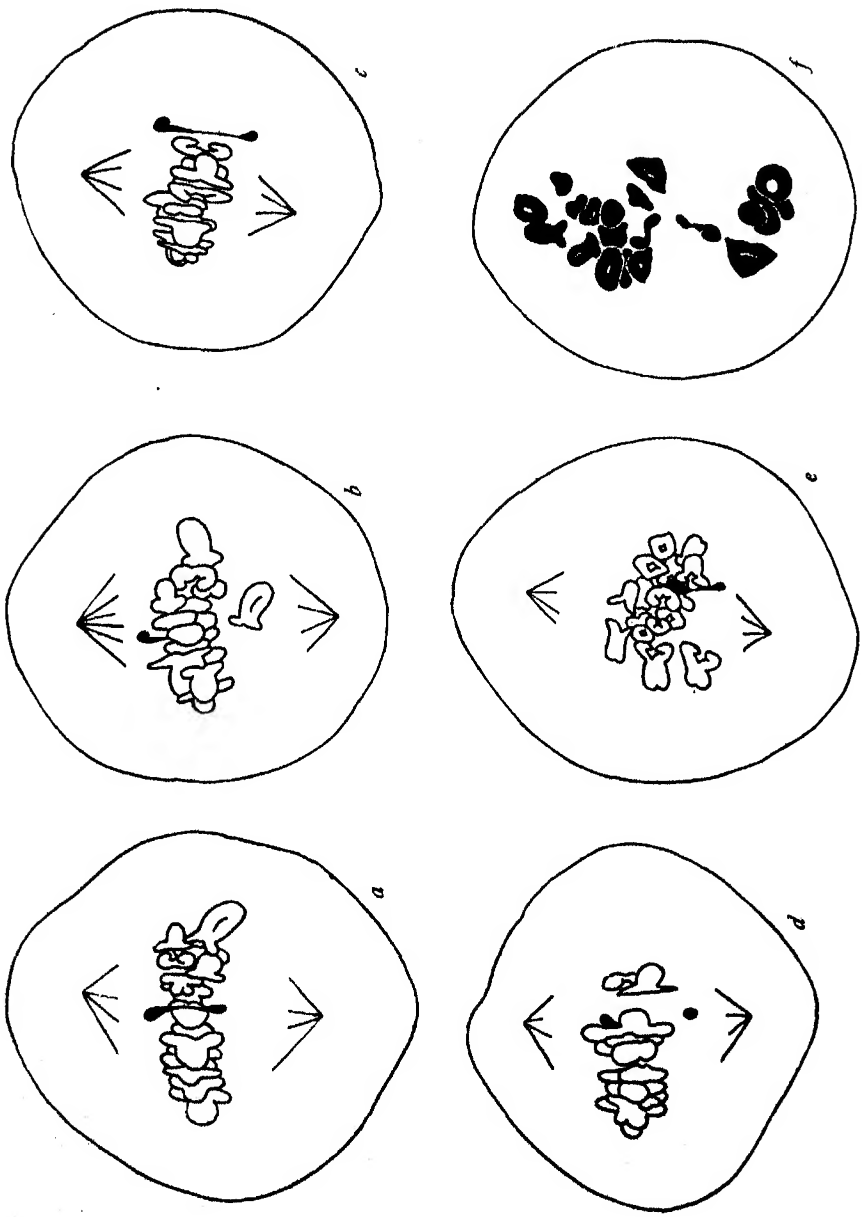


FIG. 8.—*a, b, c, d, e*, metaphase of meiosis in five spermatocytes showing the position of the asymmetrical sex bivalent; *f*, polar view of metaphase, the XY bivalent lies at the centre of the plate. ×3500.

contraction of the sex-chromosomes becomes increased and they sometimes show differential staining. The sex-bivalent lies at the periphery of the spindle during metaphase and is readily distinguished by reason of its asymmetry (fig. 8, *a*, *b*, *c*, *d*, *e*, *f*, and Pl., F, G, K). Since only this asymmetrical type of sex-bivalent was encountered, pre-reduction is regarded as being obligatory in the boar. The appearance and behaviour of the sex-chromosome during mitosis and meiosis seem to indicate that their homologous or pairing segments lie on one side of the centromere, with the other segment carrying the sex-differentiators on the opposite side in the case of the X, this segment being absent in the Y (fig. 9). In this respect the boar is in agreement with the mole (Koller, 1936). Differentiation of this kind indicates that in the X there are genes which, because they are in the differential segment, will exhibit the usual sex-linked mode of transmission, and others which, being resident in the pairing segment, will necessarily exhibit only partial sex-linkage. The pairing segment in the X of the boar is two-thirds of the total length of the chromosome. It follows therefore that according to the relative lengths of pairing and differential segments respectively, partial sex-linkage should be more frequent than complete sex-linkage in the pig.

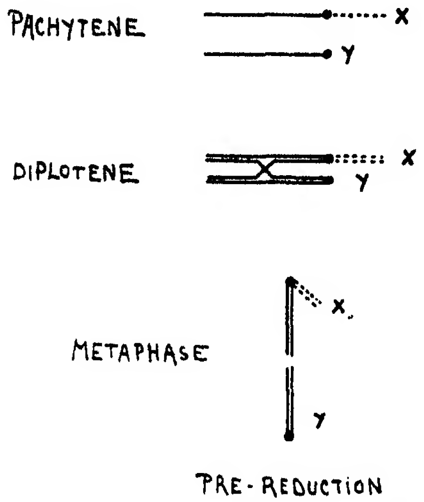


FIG. 9.—Diagram illustrating the structure of the sex-chromosomes. The pairing segment is solid, the differential segment is indicated by dotted lines.

STRUCTURAL HYBRIDITY.

Chromatid bridges were seen at the first and second meiotic anaphase in one of the animals (fig. 10, *a*, *b*, *c*, *d*, and Pl., H, I). Since it is known that such bridges and acentric fragments result from the formation of dicentric chromatids through crossing-over in a relatively inverted segment (fig. 11), it is assumed that this individual was heterozygous for an inversion in at least one of its chromosomes. The occurrence of a proximal chiasma disparate to the one in the inversion gives rise to a second anaphase bridge (Upcott, 1937). In an attempt to determine the approximate position of the chromosome region that was inverted, the frequency of the second and first anaphase bridges was examined (Table III).

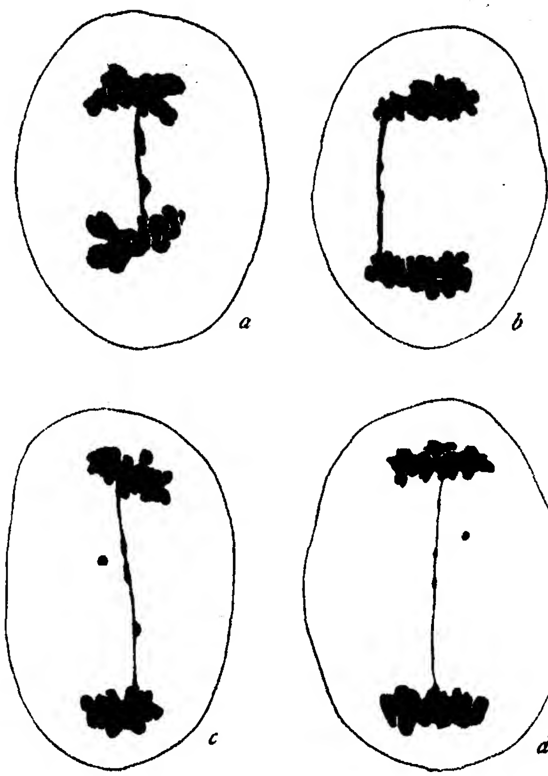


FIG. 10.—*a*, *b*, chromatid bridges at first meiotic anaphase; the fragments are not shown; *c*, *d*, bridges and fragments at second anaphase of meiosis. × 3500.

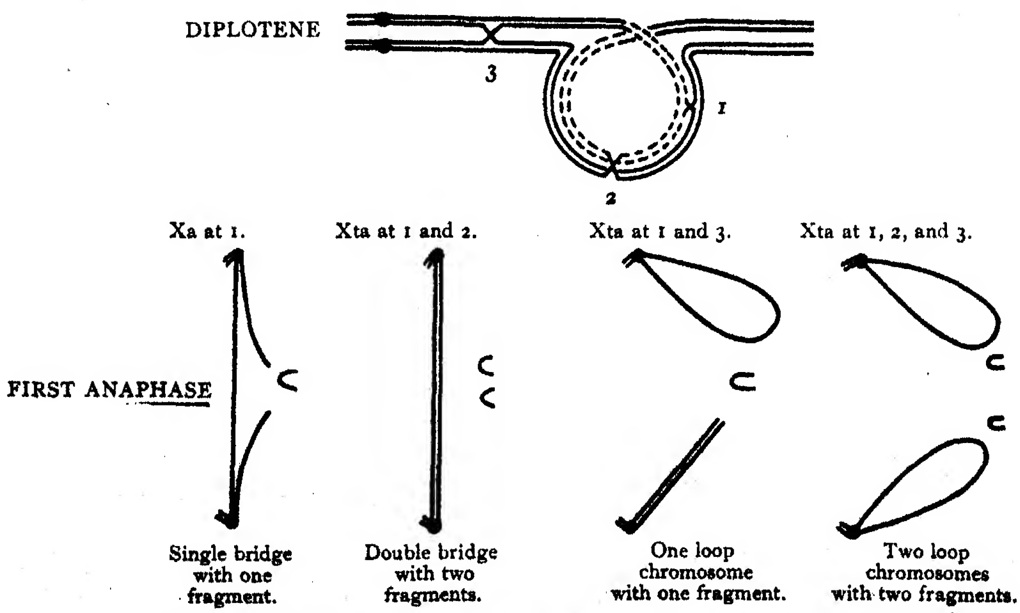


FIG. 11.—Diagram illustrating the origin of first and second meiotic anaphase bridges and fragments. The type and time of appearance of bridges depends on the number and position of chiasmata. Loop chromosomes form bridges at second anaphase.

TABLE III.—THE FREQUENCY OF FIRST AND SECOND ANAPHASE BRIDGES.

Total Cells.	First Meiotic Anaphase.		Total Cells.	Second Meiotic Anaphase.	
	Normal.	Bridge.		Normal.	Bridge.
149	134	15 (10 per cent.)	190	162	28 (14·7 per cent.)

The relative frequency of second anaphase bridges is 29 per cent. ($14{\cdot}7 \times 2$), it is higher than that of the first. Owing to the fact that chiasma formation is not at random (at least in the larger bivalents), the position and size of the relatively inverted segment could not be determined. It may involve either a short proximal or distal region of a chromosome.

SUMMARY.

1. The diploid chromosome number in the spermatogonial cells is 38. The largest chromosome is $7{\cdot}5\ \mu$ long, the smallest is $1{\cdot}5$–$2\ \mu$. The rest form a graded series between these extremes.

2. There are 5 chromosome pairs with median, 4 with submedian, and 9 with subterminal centromeres.

3. The two sex chromosomes are unequal in size; the Y being $\frac{2}{3}$ of the total length of the X chromosome. The position of the centromere in the X is submedian or subterminal, in the Y it is terminal or nearly so.

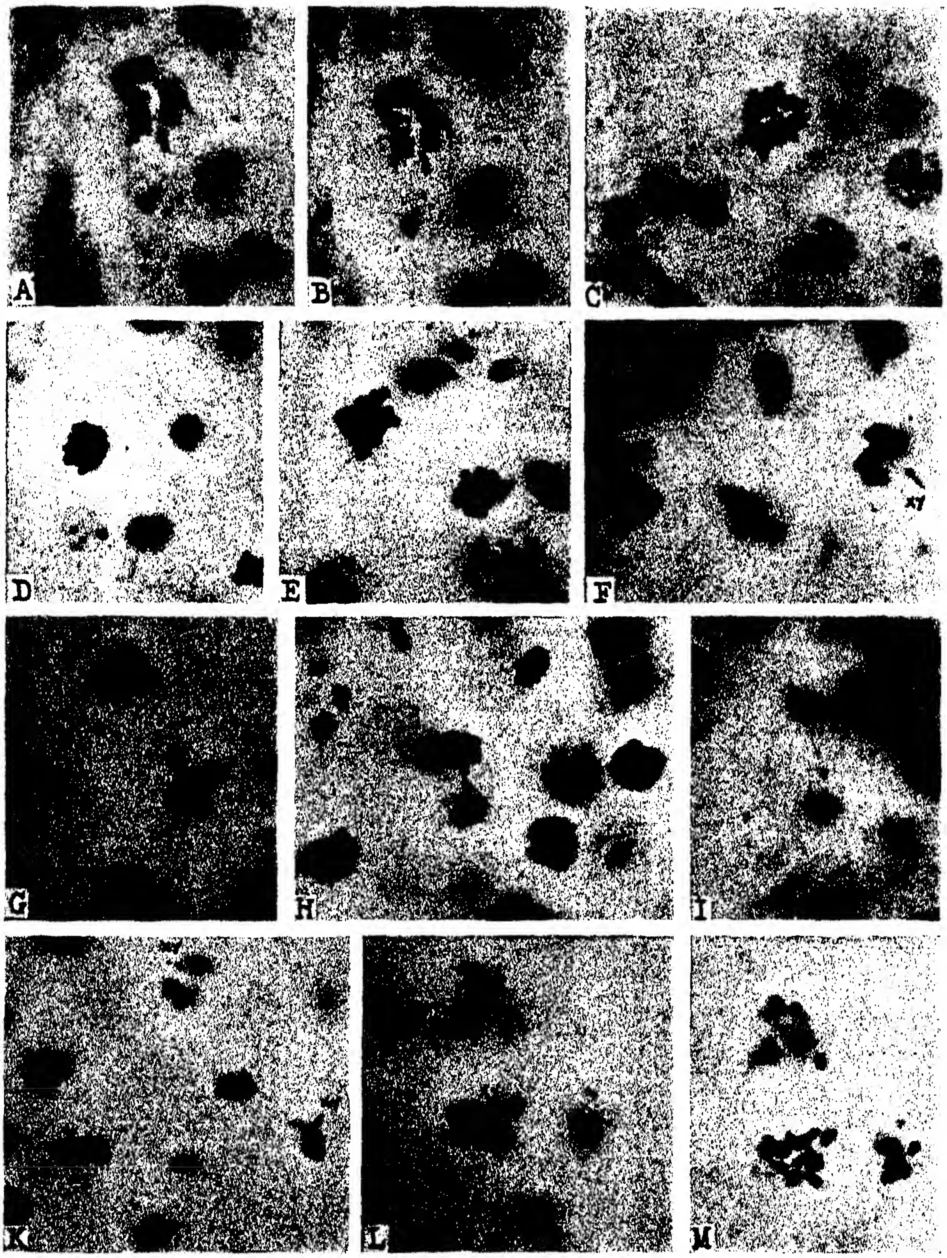
4. The number of chiasmata per nucleus at mid-diplotene and metaphase of meiosis is the same and there is very little if any terminalisation. It is assumed therefore that the number of chiasmata in the nucleus at all stages represents the number of initial cross-overs.

5. It is possible that in at least some bivalents there exists a direct proportionality between the number of chiasmata and the length of the chromosome.

6. Chiasma formation is apparently localised in the five larger bivalents, and consequently gene recombination in these bivalents is of a special kind and is limited. Those genes which are located between two adjacent chiasmata may be excluded entirely from recombination.

7. The type of sex bivalent during metaphase of meiosis is invariably asymmetrical, hence pre-reduction is obligatory. The structure of the sex bivalent suggests that the centromere lies between the pairing and differential segments.

8. In one individual a chromosome pair was heterozygous for an inversion.


A
B
C
D
E
F
G
H
I
K
L
M

REFERENCES TO LITERATURE.

DARLINGTON, C. D., 1937. *Recent Advances in Cytology*, 2nd ed., London.

—— 1939. *Evolution of the Genetic System*, Cambridge Univ. Press.

HANCE, R. T., 1917. "The Diploid Chromosome Complexes of the Pig (*Sus scrofa*) and their Variations," *Journ. Morphol.*, vol. xxx, pp. 155–201.

HILLEBRAND, P., 1936. "Untersuchungen uber die Chromosomen bei drei verschiedenen Rassen von Hausschweinen," *Dissertation*, Friedrich Wilhelms Universität, Breslau.

KRALLINGER, H. F., 1931. "Cytologische Studien an einigen Haussaugetieren," *Arch. Tierernähr. Tierz.*, vol. v, pp. 127–187.

KOLLER, P. C., 1936. "Chromosome Behaviour in the Male Ferret and Mole during Anœstrus," *Proc. Roy. Soc.*, B, vol. cxxi, pp. 192–206.

UPCOTT, M., 1937. "The Genetic Structure of *Tulipa*, II. Structural Hybridity," *Journ. Genet.*, vol. xxxiv, pp. 339–398.

WODSEDALEK, J. E., 1913. "Spermatogenesis of the Pig with Special Reference to the Accessory Chromosomes," *Biol. Bull.*, vol. xxv, pp. 8–32.

DESCRIPTION OF PLATE.

A and B. Microphotographs of metaphase plate in a spermatogonial cell taken at different focal length.

C. Mitotic metaphase, the small chromosomes are at the centre.

D. Polar view of first meiotic metaphase.

E. Polar view of first meiotic metaphase. One nucleus showing pachytene stage; the sex bivalent is indicated.

F, G, and K. Side view of meiotic metaphases showing the sex bivalent.

H. First meiotic anaphase bridge.

I. Chromatid bridge and fragment at the second meiotic anaphase.

L. Late diplotene stage showing the sex bivalent.

M. Diplotene and diakinesis stage, one ring-shaped bivalent is shown distinctly.

(*Issued separately June* 27, 1939.)

XVII.—Relations between the Elliptic Cylinder Functions. By E. L. Ince, D.Sc.

(MS. received October 31, 1938. Revised MS. received April 24, 1939. Read March 6, 1939.)

1. Introduction.

The elliptic cylinder functions, or solutions of the Mathieu equation

$$\frac{d^2y}{dx^2}+(a-2\theta\cos 2x)y=0$$

that have the period π or 2π, are of four distinct species. These are denoted respectively by $\mathrm{ce}_{2n}(x, \theta)$, $\mathrm{se}_{2n+1}(x, \theta)$, $\mathrm{ce}_{2n+1}(x, \theta)$, $\mathrm{se}_{2n+2}(x, \theta)$, where n indicates the number of zeros in the open interval $0 < x < \frac{1}{2}\pi$. The object of this paper is to show that a function of any type can be expressed in terms of functions of other types, or of their derivatives. This provides the analogies to such trigonometrical relations as

$$\cos 2nx=\cos(2n+1)x\cos x+\sin(2n+1)x\sin x,$$

but the single terms on the right-hand side of this and similar relations are replaced, in the results to be proved, by infinite series. These results are quite distinct from Whittaker's recurrence-relations (*Journ. London Math. Soc.*, vol. iv, 1929, pp. 88–96); in particular, they do not involve the non-periodic second solutions.

The paper is based on the fact that the elliptic cylinder functions satisfy homogeneous integral equations of the type

$$y(x)=\lambda\int_0^{\pi}K(x, t)y(t)dt,$$

in which the form of the nucleus $K(x, t)$ differs according to the species of the function in question, while to each species of function corresponds an infinite set of distinct nuclei. Conversely, every such nucleus can be developed as a series of products of elliptic cylinder functions of one particular species (its characteristic functions).

For the purposes of this paper a notation fuller than that hitherto used will be necessary. The Fourier series that represent the four species of functions will be written as follows *:—

* Throughout the paper, summation extends over all positive integral values of the suffix, including zero when relevant.

$$\text{ce}_{2n}(x) = \sum_r A_{2r}^{(n)} \cos 2rx, \quad . \quad . \quad . \quad . \quad (1.1)$$

$$\text{ce}_{2n+1}(x) = \sum_r A_{2r+1}^{(n)} \cos (2r+1)x, \quad . \quad . \quad . \quad (1.2)$$

$$\text{se}_{2n+1}(x) = \sum_r B_{2r+1}^{(n)} \sin (2r+1)x, \quad . \quad . \quad . \quad (1.3)$$

$$\text{se}_{2n+2}(x) = \sum_r B_{2r+2}^{(n)} \sin (2r+2)x. \quad . \quad . \quad . \quad (1.4)$$

The coefficients A and B are, of course, functions of θ, but as θ is to be regarded as the same throughout the paper, no fuller symbolism, such as $A_{2r}^{(n)}(\theta)$, will be necessary.

Since any elliptic cylinder function $y(x)$ is defined to be such that

$$\int_0^{2\pi} \{y(x)\}^2 dx = \pi,$$

we have, for every n,

$$2\{A_0^{(n)}\}^2 + \sum_r \{A_{2r}^{(n)}\}^2 = \sum_r \{A_{2r+1}^{(n)}\}^2 = \sum_r \{B_{2r+1}^{(n)}\}^2 = \sum_r \{B_{2r+2}^{(n)}\}^2 = 1.$$

2. The Eight Primary Nuclei and their Developments.

With the above notation, it may readily be verified that, if θ is positive and $k^2 = 4\theta$, the eight simplest nuclei, which are those commonly studied, and which will be termed the primary nuclei, may be developed in terms of the appropriate species of elliptic cylinder function as follows:—

$$\cos (k \cos x \cos t) = 2\sum \frac{A_0^{(n)}}{\text{ce}_{2n}(\frac{1}{2}\pi)} \text{ce}_{2n}(x)\text{ce}_{2n}(t), \quad . \quad . \quad (2.1)$$

$$\sin (k \cos x \cos t) = -k\sum \frac{A_1^{(n)}}{\text{ce}'_{2n+1}(\frac{1}{2}\pi)} \text{ce}_{2n+1}(x)\text{ce}_{2n+1}(t), \quad (2.2)$$

$$\cosh (k \sin x \sin t) = 2\sum \frac{A_0^{(n)}}{\text{ce}_{2n}(0)} \text{ce}_{2n}(x)\text{ce}_{2n}(t), \quad . \quad . \quad (2.3)$$

$$\sinh (k \sin x \sin t) = k\sum \frac{B_1^{(n)}}{\text{se}'_{2n+1}(0)} \text{se}_{2n+1}(x)\text{se}_{2n+1}(t), \quad . \quad (2.4)$$

$$\sin x \sin t \cos (k \cos x \cos t) = \sum \frac{B_1^{(n)}}{\text{se}_{2n+1}(\frac{1}{2}\pi)} \text{se}_{2n+1}(x)\text{se}_{2n+1}(t), \quad . \quad (2.5)$$

$$\sin x \sin t \sin (k \cos x \cos t) = -\tfrac{1}{2}k\sum \frac{B_2^{(n)}}{\text{se}'_{2n+2}(\frac{1}{2}\pi)} \text{se}_{2n+2}(x)\text{se}_{2n+2}(t), \quad (2.6)$$

$$\cos x \cos t \cosh (k \sin x \sin t) = \sum \frac{A_1^{(n)}}{\text{ce}_{2n+1}(0)} \text{ce}_{2n+1}(x)\text{ce}_{2n+1}(t), \quad . \quad (2.7)$$

$$\cos x \cos t \sinh (k \sin x \sin t) = \tfrac{1}{2}k\sum \frac{B_2^{(n)}}{\text{se}'_{2n+2}(0)} \text{se}_{2n+2}(x)\text{se}_{2n+2}(t). \quad . \quad (2.8)$$

The modifications required by a negative θ are obvious.

When $t=\frac{1}{2}\pi$, (2.1) becomes

$$\tfrac{1}{2}=\Sigma A_0^{(n)}\mathrm{ce}_{2n}(x).$$

If (2.2) is differentiated with respect to t, the result becomes, when $t=\frac{1}{2}\pi$,

$$\cos x=\Sigma A_1^{(n)}\mathrm{ce}_{2n+1}(x).$$

Similarly it may be shown that

$$\sin x=\Sigma B_1^{(n)}\mathrm{se}_{2n+1}(x),$$
$$\sin 2x=\Sigma B_2^{(n)}\mathrm{se}_{2n+2}(x).$$

From these, by the use of (1.1)–(1.4), it follows that

$$\Sigma\{A_0^{(n)}\}^2=\tfrac{1}{2},\qquad \Sigma\{A_1^{(n)}\}^2=\Sigma\{B_1^{(n)}\}^2=\Sigma\{B_2^{(n)}\}^2=1.$$

It is easy to generalise these results; in fact, the orthogonal properties of the functions involved enable us to show that

$$\cos 2rx=\sum_n A_{2r}^{(n)}\mathrm{ce}_{2n}(x),$$
$$\cos(2r+1)x=\sum_n A_{2r+1}^{(n)}\mathrm{ce}_{2n+1}(x),$$
$$\sin(2r+1)x=\sum_n B_{2r+1}^{(n)}\mathrm{se}_{2n+1}(x),$$
$$\sin(2r+2)x=\sum_n B_{2r+2}^{(n)}\mathrm{se}_{2n+2}(x),$$

relations which are reciprocal to (1.1)–(1.4). From these we deduce that

$$\sum_n\{A_{2r}^{(n)}\}^2=\sum_n\{A_{2r+1}^{(n)}\}^2=\sum_n\{B_{2r+1}^{(n)}\}^2=\sum_n\{B_{2r+2}^{(n)}\}^2=1$$

($r>0$ in the first expression). Likewise

$$\sum_n\{A_{2r}^{(n)}A_{2s}^{(n)}\}=0,$$

when $r\neq s$, and three similar expressions. These relations are of very great value in the numerical computation of the coefficients.

3. Relations between Functions whose Orders are of Opposite Parity.

By multiplying (2.1) through by $\sin x\sin t$ and identifying the right-hand member with that of (2.5), we have

$$\sum\frac{B_1^{(n)}}{\mathrm{se}_{2n+1}(\frac{1}{2}\pi)}\mathrm{se}_{2n+1}(x)\mathrm{se}_{2n+1}(t)=2\sin x\sin t\sum\frac{A_0^{(n)}}{\mathrm{ce}_{2n}(\frac{1}{2}\pi)}\mathrm{ce}_{2n}(x)\mathrm{ce}_{2n}(t).\quad(3.1)$$

If this equation is multiplied throughout by $se_{2m+1}(t)$ and then integrated with respect to t over the range $(0, 2\pi)$, a single term will remain on the left on account of the orthogonality of $se_{2m+1}(t)$ with $se_{2n+1}(t)$ when $n \neq m$. Thus

$$\frac{B_1^{(m)}}{se_{2m+1}(\frac{1}{2}\pi)} se_{2m+1}(x) = \frac{2 \sin x}{\pi} \sum_n \frac{A_0^{(n)}}{ce_{2n}(\frac{1}{2}\pi)} ce_{2n}(x) \int_0^{2\pi} ce_{2n}(t) se_{2m+1}(t) \sin t \, dt$$

$$= \sin x \sum_n \frac{A_0^{(n)}}{ce_{2n}(\frac{1}{2}\pi)} \left[\sum_r A_{2r}^{(n)} \{ B_{2r+1}^{(m)} - B_{2r-1}^{(m)} \} \right] ce_{2n}(x). \quad . \quad (3.2)$$

In the same way it may be shown that

$$\frac{B_2^{(m)}}{se'_{2m+2}(\frac{1}{2}\pi)} se_{2m+2}(x) = \sin x \sum_n \frac{A_1^{(n)}}{ce'_{2n+1}(\frac{1}{2}\pi)} \left[\sum_r A_{2r+1}^{(n)} \{ B_{2r+2}^{(m)} - B_{2r}^{(m)} \} \right] ce_{2n+1}(x), \quad (3.3)$$

$$\frac{A_1^{(m)}}{ce_{2m+1}(0)} ce_{2m+1}(x) = \cos x \sum_n \frac{A_0^{(n)}}{ce_{2n}(0)} \left[\sum_r A_{2r}^{(n)} \{ A_{2r+1}^{(m)} + A_{2r-1}^{(m)} \} \right] ce_{2n}(x), \quad . \quad (3.4)$$

$$\frac{B_2^{(m)}}{se'_{2m+2}(0)} se_{2m+2}(x) = \cos x \sum_n \frac{B_1^{(n)}}{se'_{2n+1}(0)} \left[\sum_r B_{2r+1}^{(n)} \{ B_{2r+2}^{(m)} + B_{2r}^{(m)} \} \right] se_{2n+1}(x). \quad (3.5)$$

Equation (3.2) is the generalisation of the trigonometrical relation by which $\sin (2m+1)x/\sin x$ is expressed as a series of cosines of even multiples of x; (3.3)–(3.5) are similar generalisations.

Now if (3.1) is multiplied by $ce_{2m}(t)/\sin t$ and the resulting expression integrated with respect to t over the range $(0, 2\pi)$, we obtain

$$2\pi \sin x \frac{A_0^{(m)}}{ce_{2m}(\frac{1}{2}\pi)} ce_{2m}(x) = \sum \frac{B_1^{(n)}}{se_{2n+1}(\frac{1}{2}\pi)} se_{2n+1}(x) \int_0^{2\pi} \frac{se_{2n+1}(t) ce_{2m}(t)}{\sin t} dt. \quad (3.6)$$

Evaluating the integral with the aid of (3.2), we finally arrive at the relation

$$\sin x \, ce_{2m}(x) = \tfrac{1}{2} \sum_n \left[\sum_r B_{2r+1}^{(n)} \{ A_{2r}^{(m)} - A_{2r+2}^{(m)} \} \right] se_{2n+1}(x). \quad . \quad (3.7)$$

This is a generalisation of

$$\sin x \cos 2mx = \tfrac{1}{2} \{ \sin (2m+1)x - \sin (2m-1)x \};$$

expressions for $\sin x \, ce_{2m+1}(x)$, $\cos x \, ce_{2m}(x)$, and $\cos x \, se_{2m+1}(x)$ may be established in much the same way. They may also be deduced by a direct application of the orthogonal properties of the elliptic cylinder functions.

The relation

$$\cos x\, \mathrm{ce}_{2m+1}(x) = \tfrac{1}{2}\sum_n \left[\sum_r A_{2r}^{(n)}\{A_{2r+1}^{(m)} + A_{2r-1}^{(m)}\}\right] \mathrm{ce}_{2n}(x), \qquad (3.8)$$

and a similar expression for $\sin x\, \mathrm{se}_{2m+1}(x)$, may also be established.

4. Relations between Functions whose Orders are of Like Parity.

Differentiating (2.1) with respect to t and using (2.6) on the left-hand side of the result, we obtain the relation

$$4 \sin x \sum \frac{A_0^{(n)}}{\mathrm{ce}_{2n}(\tfrac{1}{2}\pi)} \mathrm{ce}_{2n}(x)\mathrm{ce}'_{2n}(t) = -k^2 \cos x \sum \frac{B_2^{(n)}}{\mathrm{se}'_{2n+2}(\tfrac{1}{2}\pi)} \mathrm{se}_{2n+2}(x)\mathrm{se}_{2n+2}(t), \quad (4.1)$$

from which we deduce that

$$\begin{aligned} \frac{k^2 B_2^{(m)}}{\mathrm{se}'_{2m+2}(\tfrac{1}{2}\pi)} \mathrm{se}_{2m+2}(x) &= -\frac{4}{\pi} \tan x \sum \frac{A_0^{(n)}}{\mathrm{ce}_{2n}(\tfrac{1}{2}\pi)} \mathrm{ce}_{2n}(x) \int_0^{2\pi} \mathrm{ce}'_{2n}(t)\mathrm{se}_{2m+2}(t)dt \\ &= 8 \tan x \sum_n \frac{A_0^{(n)}}{\mathrm{ce}_{2n}(\tfrac{1}{2}\pi)} \left[\sum_r r A_{2r}^{(n)} B_{2r}^{(m)}\right] \mathrm{ce}_{2n}(x); \end{aligned} \qquad (4.2)$$

and in similar fashion it may be proved that

$$\frac{B_1^{(m)}}{\mathrm{se}_{2m+1}(\tfrac{1}{2}\pi)} \mathrm{se}_{2m+1}(x) = -\tan x \sum_n \frac{A_1^{(n)}}{\mathrm{ce}'_{2n+1}(\tfrac{1}{2}\pi)} \left[\sum_r (2r+1) A_{2r+1}^{(n)} B_{2r+1}^{(m)}\right] \mathrm{ce}_{2n+1}(x), \quad (4.3)$$

$$\frac{k^2 B_2^{(m)}}{\mathrm{se}'_{2m+2}(0)} \mathrm{se}_{2m+2}(x) = -8 \cot x \sum_n \frac{A_0^{(n)}}{\mathrm{ce}_{2n}(0)} \left[\sum_r r A_{2r}^{(n)} B_{2r}^{(m)}\right] \mathrm{ce}_{2n}(x), \qquad (4.4)$$

$$\frac{A_1^{(m)}}{\mathrm{ce}_{2m+1}(0)} \mathrm{ce}_{2m+1}(x) = \cot x \sum_n \frac{B_1^{(n)}}{\mathrm{se}'_{2n+1}(0)} \left[\sum_r (2r+1) B_{2r+1}^{(n)} A_{2r+1}^{(m)}\right] \mathrm{se}_{2n+1}(x). \qquad (4.5)$$

To these may be added relations that can be deduced directly from the orthogonal properties of the functions, as, for instance,

$$\begin{aligned} \sin 2x\, \mathrm{ce}_{2m}(x) &= \frac{1}{\pi} \sum \mathrm{se}_{2n+2}(x) \int_0^{2\pi} \mathrm{se}_{2n+2}(t)\mathrm{ce}_{2m}(t) \sin 2t\, dt \\ &= \tfrac{1}{2}\sum_n \left[\sum_r B_{2r}^{(n)}\{A_{2r-2}^{(m)} - A_{2r+2}^{(m)}\}\right] \mathrm{se}_{2n+2}(x). \end{aligned} \qquad (4.6)$$

5. The Derivatives.

If we differentiate (2.1) with respect to x and make use of (2.2) we obtain

$$2 \sum \frac{A_0^{(n)}}{\mathrm{ce}_{2n}(\tfrac{1}{2}\pi)} \mathrm{ce}'_{2n}(x)\mathrm{ce}_{2n}(t) = -k^2 \sin x \cos t \sum \frac{A_1^{(n)}}{\mathrm{ce}'_{2n+1}(\tfrac{1}{2}\pi)} \mathrm{ce}_{2n+1}(x)\mathrm{ce}_{2n+1}(t). \quad (5.1)$$

Hence, multiplying by $ce_{2m}(t)$ and integrating, we arrive at the following expression for the derivative $ce'_{2m}(x)$:—

$$\frac{A_0^{(m)}}{ce_{2m}(\frac{1}{2}\pi)}ce'_{2m}(x) = -\frac{k^2}{2\pi}\sin x\sum\frac{A_1^{(n)}}{ce'_{2n+1}(\frac{1}{2}\pi)}ce_{2n+1}(x)\int_0^{2\pi}ce_{2n+1}(t)ce_{2m}(t)\cos t\,dt$$

$$= -\tfrac{1}{4}k^2\sin x\sum_n\frac{A_1^{(n)}}{ce'_{2n+1}(\frac{1}{2}\pi)}\left[\sum_r A_{2r+1}^{(n)}\{A_{2r}^{(m)}+A_{2r+2}^{(m)}\}\right]ce_{2n+1}(x). \quad (5.2)$$

In a similar way it may be proved that

$$\frac{A_1^{(m)}}{ce'_{2m+1}(\frac{1}{2}\pi)}ce'_{2m+1}(x) = \sin x\sum_n\frac{A_0^{(n)}}{ce_{2n}(\frac{1}{2}\pi)}\left[\sum_r A_{2r}^{(n)}\{A_{2r-1}^{(m)}+A_{2r+1}^{(m)}\}\right]ce_{2n}(x), \quad (5.3)$$

$$\frac{A_0^{(m)}}{ce_{2m}(0)}ce'_{2m}(x) = \tfrac{1}{4}k^2\cos x\sum_n\frac{B_1^{(n)}}{se'_{2n+1}(0)}\left[\sum_r B_{2r+1}^{(n)}\{A_{2r}^{(m)}-A_{2r+2}^{(m)}\}\right]se_{2n+1}(x), \quad (5.4)$$

$$\frac{B_1^{(m)}}{se'_{2m+1}(0)}se'_{2m+1}(x) = \cos x\sum_n\frac{A_0^{(n)}}{ce_{2n}(0)}\left[\sum_r A_{2r}^{(n)}\{B_{2r+1}^{(m)}-B_{2r-1}^{(m)}\}\right]ce_{2n}(x). \quad (5.5)$$

These are evidently closely allied to (3.2)–(3.5).

A direct application of the orthogonal properties of the elliptic cylinder functions will lead to a distinct set of relations, of which the following are typical examples:—

$$ce'_{2m+1}(x) = -\sum_n\left[\sum_r(2r+1)B_{2r+1}^{(n)}A_{2r+1}^{(m)}\right]se_{2n+1}(x), \quad (5.6)$$

$$se'_{2m+2}(x) = 2\sum_n\left[\sum_r rA_{2r}^{(n)}B_{2r}^{(m)}\right]ce_{2n}(x). \quad (5.7)$$

Other relations result from differentiating (2.5)–(2.8); for example, if (2.8) is differentiated and (2.4) and (2.7) used in the result, we have a relation from which it may be deduced that

$$\frac{B_2^{(m)}}{se'_{2m+2}(0)}se'_{2m+2}(x) = \frac{2\cos x}{\pi}\sum\frac{A_1^{(n)}}{ce_{2n+1}(0)}ce_{2n+1}(x)\int_0^{2\pi}ce_{2n+1}(t)se_{2m+2}(t)\sin t\,dt$$

$$-\frac{2\sin x}{\pi}\sum\frac{B_1^{(n)}}{se'_{2n+1}(0)}se_{2n+1}(x)\int_0^{2\pi}se_{2n+1}(t)se_{2m+2}(t)\cos t\,dt$$

$$= \cos x\sum_n\frac{A_1^{(n)}}{ce_{2n+1}(0)}\left[\sum_r A_{2r+1}^{(n)}\{B_{2r+2}^{(m)}-B_{2r}^{(m)}\}\right]ce_{2n+1}(x)$$

$$-\sin x\sum_n\frac{B_1^{(n)}}{se'_{2n+1}(0)}\left[\sum_r B_{2r+1}^{(n)}\{B_{2r+2}^{(m)}+B_{2r}^{(m)}\}\right]se_{2n+1}(x). \quad (5.8)$$

This is one of the formulæ in elliptic cylinder functions generalising the trigonometrical addition theorem

$$\cos(2m+2)x = \cos x\cos(2m+1)x - \sin x\sin(2m+1)x;$$

others will be found in the following section.

6. Secondary Nuclei and their Developments.

Starting with each of the eight primary nuclei, we can obtain an infinite set of nuclei with the same characteristic functions in the following manner. If $K_n(x, t)$ is a nucleus and Δ denotes the operator

$$\frac{\partial^2}{\partial t^2} - k^2 \cos^2 t,$$

then $\Delta K_n(x, t)$ is also a nucleus (Ince, *Proc. Roy. Soc. Edin.*, vol. xliv, 1924, pp. 242–247). We define $K_{n+1}(x, t)$ as $\Delta K_n(x, t)$ modified by the removal of any terms which are constant multiples of $K_n(x, t)$, and by the omission of any irrelevant constant factor from what remains. Thus, starting from a primary nucleus $K_1(x, t)$, we obtain the set of nuclei $K_n(x, t)$ for $n = 2, 3, 4, \ldots$

For example, the two secondary nuclei whose characteristic functions are $\mathrm{ce}_{2n}(x)$, with their developments in terms of the characteristic functions, are

$$\cos x \cos t \sin (k \cos x \cos t) + k \sin^2 x \sin^2 t \cos (k \cos x \cos t)$$
$$= \tfrac{1}{2}k \sum \frac{2A_0^{(n)} - A_2^{(n)}}{\mathrm{ce}_{2n}(\tfrac{1}{2}\pi)} \mathrm{ce}_{2n}(x)\mathrm{ce}_{2n}(t), \quad (6.1)$$

$$\sin x \sin t \sinh (k \sin x \sin t) + k \cos^2 x \cos^2 t \cosh (k \sin x \sin t)$$
$$= \tfrac{1}{2}k \sum \frac{2A_0^{(n)} + A_2^{(n)}}{\mathrm{ce}_{2n}(0)} \mathrm{ce}_{2n}(x)\mathrm{ce}_{2n}(t). \quad (6.2)$$

Making use of (2.2) and (2.5) on the left-hand side of (6.1), we transform that relation into

$$\sum \frac{2A_0^{(n)} - A_2^{(n)}}{\mathrm{ce}_{2n}(\tfrac{1}{2}\pi)} \mathrm{ce}_{2n}(x)\mathrm{ce}_{2n}(t) = 2 \sin x \sin t \sum \frac{B_1^{(n)}}{\mathrm{se}_{2n+1}(\tfrac{1}{2}\pi)} \mathrm{se}_{2n+1}(x)\mathrm{se}_{2n+1}(t)$$
$$- 2 \cos x \cos t \sum \frac{A_1^{(n)}}{\mathrm{ce}'_{2n+1}(\tfrac{1}{2}\pi)} \mathrm{ce}_{2n+1}(x)\mathrm{ce}_{2n+1}(t),$$

whence we obtain by the usual procedure

$$\frac{2A_0^{(m)} - A_2^{(m)}}{\mathrm{ce}_{2m}(\tfrac{1}{2}\pi)} \mathrm{ce}_{2m}(x) = \frac{2 \sin x}{\pi} \sum \frac{B_1^{(n)}}{\mathrm{se}_{2n+1}(\tfrac{1}{2}\pi)} \mathrm{se}_{2n+1}(x) \int_0^{2\pi} \mathrm{se}_{2n+1}(t)\mathrm{ce}_{2m}(t) \sin t \, dt$$
$$- \frac{2 \cos x}{\pi} \sum \frac{A_1^{(n)}}{\mathrm{ce}'_{2n+1}(\tfrac{1}{2}\pi)} \mathrm{ce}_{2n+1}(x) \int_0^{2\pi} \mathrm{ce}_{2n+1}(t)\mathrm{ce}_{2m}(t) \cos t \, dt$$
$$= \sin x \sum_n \frac{B_1^{(n)}}{\mathrm{se}_{2n+1}(\tfrac{1}{2}\pi)} \Big[\sum_r B_{2r+1}^{(n)} \{A_{2r}^{(m)} - A_{2r+2}^{(m)}\} \Big] \mathrm{se}_{2n+1}(x)$$
$$- \cos x \sum_n \frac{A_1^{(n)}}{\mathrm{ce}'_{2n+1}(\tfrac{1}{2}\pi)} \Big[\sum_r A_{2r+1}^{(n)} \{A_{2r}^{(m)} + A_{2r+2}^{(m)}\} \Big] \mathrm{ce}_{2n+1}(x). \quad (6.3)$$

In like fashion we obtain from (6.2) the similar but independent relation

$$\frac{2A_0^{(m)}+A_2^{(m)}}{\mathrm{ce}_{2m}(0)}\mathrm{ce}_{2m}(x)=\sin x\sum_n \frac{B_1^{(n)}}{\mathrm{se}_{2n+1}'(0)}\left[\sum_r B_{2r+1}^{(n)}\{A_{2r}^{(m)}-A_{2r+2}^{(m)}\}\right]\mathrm{se}_{2n+1}(x)$$
$$+\cos x\sum_n \frac{A_1^{(n)}}{\mathrm{ce}_{2n+1}(0)}\left[\sum_r A_{2r+1}^{(n)}\{A_{2r}^{(m)}+A_{2r+2}^{(m)}\}\right]\mathrm{ce}_{2n+1}(x). \quad (6.4)$$

These are further generalisations of the addition theorem for the cosine.

Numerous relations of similar form can be obtained by using the other secondary nuclei.

7. Integral Relations.

It will be found that to every relation between elliptic cylinder functions of two species there corresponds a relation between two definite integrals which involve these functions, and possibly also the derivative of one of them. For example, if we multiply the first equality of (3.2) through by $\mathrm{ce}_{2n}(x)/\sin x$ and integrate over the range $(0, 2\pi)$ we obtain

$$\frac{B_1^{(m)}}{\mathrm{se}_{2m+1}(\frac{1}{2}\pi)}\int_0^{2\pi}\frac{\mathrm{ce}_{2n}(t)\mathrm{se}_{2m+1}(t)}{\sin t}dt=\frac{A_0^{(n)}}{\mathrm{ce}_{2n}(\frac{1}{2}\pi)}\int_0^{2\pi}\mathrm{ce}_{2n}(t)\mathrm{se}_{2m+1}(t)\sin t\,dt. \quad (7.1)$$

Similarly, the first equality of (4.2) leads to

$$\frac{k^2B_2^{(m)}}{\mathrm{se}_{2m+2}'(\frac{1}{2}\pi)}\int_0^{2\pi}\mathrm{ce}_{2n}(t)\mathrm{se}_{2m+2}(t)\cot t\,dt=-\frac{4A_0^{(n)}}{\mathrm{ce}_{2n}(\frac{1}{2}\pi)}\int_0^{2\pi}\mathrm{ce}_{2n}'(t)\mathrm{se}_{2m+2}(t)dt$$
$$=\frac{4A_0^{(n)}}{\mathrm{ce}_{2n}(\frac{1}{2}\pi)}\int_0^{2\pi}\mathrm{ce}_{2n}(t)\mathrm{se}_{2m+2}'(t)dt \quad (7.2)$$

(on integrating by parts).

Lastly, we obtain from (5.2)

$$\frac{A_0^{(m)}}{\mathrm{ce}_{2m}(\frac{1}{2}\pi)}\int_0^{2\pi}\frac{\mathrm{ce}_{2n+1}(t)\mathrm{ce}_{2m}'(t)}{\sin t}dt=-\frac{\frac{1}{4}k^2A_1^{(n)}}{\mathrm{ce}_{2n+1}'(\frac{1}{2}\pi)}\int_0^{2\pi}\mathrm{ce}_{2n+1}(t)\mathrm{ce}_{2m}(t)\cos t\,dt. \quad (7.3)$$

(*Issued separately June* 28, 1939.)

XVIII.—Tests of Significance of the Differences between Regression Coefficients derived from Two Sets of Correlated Variates.

By **F. Yates**, Sc.D., Rothamsted Experimental Station. *Communicated by* A. C. AITKEN, D.Sc., F.R.S.

(MS. received March 24, 1939. Read May 1, 1939.)

INTRODUCTION.

THE calculation of the partial regression of different, but correlated, dependent variates on the same set of independent variates is of common occurrence. Frequently, also, the differences between the resultant regression coefficients are of interest, and tests of significance for these differences are consequently required. In the classical wheat experiments on Broadbalk field at Rothamsted, for example, the influence of rainfall on yield has been investigated for the various plots, which receive different manurial treatments (Fisher, 1924). An extension of the method of partial regressions was used, which analytically amounted to the evaluation, for each plot separately, of the regression of yield on quantities representing the amount and distribution of rain in each year, these quantities being the same for all plots, since the plots were situated on the same field. One of the most interesting aspects of these results was the differences which were revealed between the effects of rainfall on the differently manured plots. (See also Cochran, 1935.)

The somewhat analogous case in which both the dependent and the independent variates are different is also of fairly frequent occurrence. In the sampling observations on the growth of wheat of the Agricultural Meteorological Scheme, for example, two varieties are observed at each of a number of stations, and the regression of yield on various earlier measurements of plant growth has been calculated for each variety separately. These measurements are themselves different for the two varieties, and consequently the tests of significance appropriate to the case in which the independent variates have the same values for each of the dependent variates requires modification. The purpose of this note is to investigate this problem.

Tests of Significance when there is only One Set of Independent Variates.

When there is only one set of independent variates the significance of the difference between the coefficients of the regressions of the different dependent variates is very simply tested. If the differences of the corresponding values, y and y', of any two dependent variates are taken, and the regression coefficients, B_1, etc., of these differences on the independent variates, x_1, etc., are calculated, the regression coefficients so obtained will be identical with the differences, $b_1 - b_1'$, etc., of the coefficients of the regressions of each variate separately. The significance of $b_1 - b_1'$, etc., may therefore be tested by testing the significance of B_1, etc., in the ordinary manner, working throughout with differences $y - y'$.

Symbolically, if the estimated regressions of y and y' are

$$Y = b_1x_1 + b_2x_2 + b_3x_3 + \dots$$
$$Y' = b_1'x_1 + b_2'x_2 + b_3'x_3 + \dots$$

then the regression of $y - y'$ is

$$Y - Y' = B_1x_1 + B_2x_2 + B_3x_3 + \dots,$$

where $B_1 = b_1 - b_1'$, etc., and the estimate of the variance of B_1, in the usual notation (Fisher, *Statistical Methods*), is

$$V(B_1) = c_{11}S(y - y' - Y + Y')^2/(n - k),$$

where n is the number of observations and k the number of independent variates.

We also have

$$S(y - y' - Y + Y')^2 = S(y - y')^2 - B_1Sx_1(y - y') - B_2Sx_2(y - y') - B_3Sx_3(y - y') - \dots$$

The formula for the variance of B_1 may also be written

$$V(B_1) = c_{11}\{V_r(y) + V_r(y') - 2\text{Cov}_r\,(yy')\},$$

the residual variances and covariance of y and y' being indicated. Their estimates are

$$\begin{aligned}(n-k)\,V_r(y) &= Sy^2 - b_1Sx_1y - b_2Sx_2y - b_3Sx_3y - \dots\\ (n-k)\,V_r(y') &= Sy'^2 - b_1'Sx_1y' - b_2'Sx_2y' - b_3'Sx_3y' - \dots\\ (n-k)\text{Cov}_r\,(yy') &= Syy' - b_1Sx_1y' - b_2Sx_2y' - b_3Sx_3y' - \dots\\ &= Syy' - b_1'Sx_1y - b_2'Sx_2y - b_3'Sx_3y - \dots\end{aligned}$$

This is the most convenient form for computation, since a table of sums of squares and products can be constructed in the ordinary analysis of variance form, the sums of products being calculated in an exactly

analogous manner to the sums of squares. The only sum of products not already calculated when evaluating the separate regressions and testing their significance is Syy'.

A case which very frequently arises in practice is that in which one of the terms of the regression equation is a constant, b_0. This can be introduced by giving x_0 the constant value unity. Group effects can be similarly eliminated by introducing a constant for each group. It can easily be shown that as far as the remaining coefficients are concerned this procedure will give the same result as that obtained by replacing all variates by their deviations from their respective group means, omitting the group constants entirely, and reducing n by the number of groups, so that n now represents the number of available degrees of freedom. Exactly the same procedure holds when dealing with two sets of correlated independent variates, provided the grouping is the same for both sets. Consequently this case need not be separately discussed.

Variance of the Difference of a Pair of Coefficients when the Independent Variates are Different.

When some or all of the independent variates have different sets of values for the different dependent variates the situation is more complex.

Let the regressions in this case (for three independent variates) be

$$Y = b_1 x_1 + b_2 x_2 + b_3 x_3,$$
$$Y' = b'_1 x'_1 + b'_2 x'_2 + b'_3 x'_3.$$

Then

$$b_1 - b'_1 = c_{11} Sx_1 y + c_{12} Sx_2 y + c_{13} Sx_3 y - c'_{11} Sx'_1 y' - c'_{12} Sx'_2 y' - c'_{13} Sx'_3 y'.$$

This is a linear function of the y's and y''s, and the variance of $b_1 - b'_1$ is consequently given by

$$\begin{aligned} V(b_1 - b'_1) &= S(c_{11}x_1 + c_{12}x_2 + c_{13}x_3)^2 V_r(y) \\ &\quad + S(c'_{11}x'_1 + c'_{12}x'_2 + c'_{13}x'_3)^2 V_r(y') \\ &\quad - 2S(c_{11}x_1 + c_{12}x_2 + c_{13}x_3)(c'_{11}x'_1 + c'_{12}x'_2 + c'_{13}x'_3) \operatorname{Cov}_r(yy') \\ &= c_{11} V_r(y) + c'_{11} V_r(y') - 2C_{11} \operatorname{Cov}_r(yy'), \end{aligned}$$

where

$$C_{11} = S(c_{11}x_1 + c_{12}x_2 + c_{13}x_3)(c'_{11}x'_1 + c'_{12}x'_2 + c'_{13}x'_3),$$

provided the residual variances and covariance of y and y' are known.

Similarly,

$$C_{22} = S(c_{12}x_1 + c_{22}x_2 + c_{23}x_3)(c'_{12}x'_1 + c'_{22}x'_2 + c'_{23}x'_3),$$

and if the covariance of $b_1 - b'_1$ and $b_2 - b'_2$ is denoted by

$$\operatorname{Cov}(b_1 - b'_1)(b_2 - b'_2) = c_{12} V_r(y) + c'_{12} V_r(y') - (C_{12} + C_{21}) \operatorname{Cov}_r(yy')$$

we have

$$C_{12} = S(c_{11}x_1 + c_{12}x_2 + c_{13}x_3)(c'_{12}x'_1 + c'_{22}x'_2 + c'_{23}x'_3),$$
$$C_{21} = S(c_{12}x_1 + c_{22}x_2 + c_{23}x_3)(c'_{11}x'_1 + c'_{12}x'_2 + c'_{13}x'_3).$$

In order to calculate the C's it is best to form two new matrices

$$\begin{matrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{matrix} \quad \text{and} \quad \begin{matrix} d'_{11} & d'_{12} & d'_{13} \\ d'_{21} & d'_{22} & d'_{23} \\ d'_{31} & d'_{32} & d'_{33} \end{matrix}$$

such that

$$d_{rs} = c_{r1}Sx_1x'_s + c_{r2}Sx_2x'_s + c_{r3}Sx_3x'_s,$$
$$d'_{rs} = c'_{1s}Sx_rx'_1 + c'_{2s}Sx_rx'_2 + c'_{3s}Sx_rx'_3.$$

The conventions here adopted should be carefully noted. If the matrix of the sums of the cross products of the x's and x''s is arranged in the form

	x'_1	x'_2	x'_3
x_1	..	..	..
x_2	..	..	..
x_3	..	..	..

then the sums of products derived from the c's and the first *column* form the first *column* of the d matrix, and so on, and the sums of products derived from the c''s and the first *row* form the first *row* of the d' matrix.

In order to calculate the d's and d''s we require a knowledge of the sums of products of each of the x's with each of the x''s.

We may now form the C matrix from the c's and the d' matrix, or the c''s and the d matrix, according to the following equation:—

$$C_{rs} = c_{r1}d'_{1s} + c_{r2}d'_{2s} + c_{r3}d'_{3s} = c'_{1s}d_{r1} + c'_{2s}d_{r2} + c'_{3s}d_{r3}.$$

The operations are the same as those adopted for the formation of the d and the d' matrices respectively.

Estimation of the Residual Covariance.

The estimation of the residual covariance of y and y' also presents difficulties.

The first step is the evaluation of the sum of the products of the residuals. We have

$$\begin{aligned} S(y-Y)(y'-Y') &= S(y - b_1x_1 - b_2x_2 - b_3x_3)(y' - b'_1x'_1 - b'_2x'_2 - b'_3x'_3) \\ &= Syy' - b_1Sx_1y' - b_2Sx_2y' - b_3Sx_3y' - b'_1Sx'_1y - b'_2Sx'_2y - b'_3Sx'_3y \\ &\quad + b_1(b'_1Sx_1x'_1 + b'_2Sx_1x'_2 + b'_3Sx_1x'_3) \\ &\quad + b_2(b'_1Sx_2x'_1 + b'_2Sx_2x'_2 + b'_3Sx_2x'_3) \\ &\quad + b_3(b'_1Sx_3x'_1 + b'_2Sx_3x'_2 + b'_3Sx_3x'_3). \end{aligned}$$

The expectation, in terms of the covariance of the true errors of y and y', of this sum must now be evaluated.

If the true values of the regression coefficients are β_1, β_2 . . . (the x's being assumed errorless) then the true errors

$$\epsilon = y - \beta_1 x_1 - \beta_2 x_2 - \beta_3 x_3$$

and

$$\epsilon' = y' - \beta_1' x_1' - \beta_2' x_2' - \beta_3' x_3'$$

are normally distributed about zero, and

$$V(\epsilon) = V_r(y), \qquad V(\epsilon') = V_r(y'), \qquad \text{Cov}\,(\epsilon\epsilon') = \text{Cov}_r\,(yy') = \kappa \text{ say.}$$

If e and e' represent the actual residuals

$$e = y - Y = y - b_1 x_1 - b_2 x_2 - b_3 x_3 = \epsilon - (b_1 - \beta_1)x_1 - (b_2 - \beta_2)x_2 - (b_3 - \beta_3)x_3,$$

and

$$\begin{aligned} b_1 - \beta_1 &= c_{11} S x_1 y + c_{12} S x_2 y + c_{13} S x_3 y - \beta_1 \\ &= (c_{11} S x_1 \epsilon + c_{12} S x_2 \epsilon + c_{13} S x_3 \epsilon) \\ &\quad + \beta_1 (c_{11} S x_1^2 + c_{12} S x_1 x_2 + c_{13} S x_1 x_3) \\ &\quad + \beta_2 (c_{11} S x_1 x_2 + c_{12} S x_2^2 + c_{13} S x_2 x_3) \\ &\quad + \beta_3 (c_{11} S x_1 x_3 + c_{12} S x_2 x_3 + c_{13} S x_3^2) - \beta_1 \\ &= c_{11} S x_1 \epsilon + c_{12} S x_2 \epsilon + c_{13} S x_3 \epsilon, \end{aligned}$$

etc.

Hence

$$\begin{aligned} e = \epsilon &- (c_{11} x_1 + c_{12} x_2 + c_{13} x_3) S x_1 \epsilon \\ &- (c_{12} x_1 + c_{22} x_2 + c_{23} x_3) S x_2 \epsilon \\ &- (c_{13} x_1 + c_{23} x_2 + c_{33} x_3) S x_3 \epsilon, \end{aligned}$$

and similarly for e'.

In order to evaluate the expectation of the sum of the products of the residuals, See', products of corresponding ϵ and ϵ' must be replaced by κ and products of non-corresponding ϵ and ϵ' by zero.

$$\begin{aligned} See' = S\epsilon\epsilon' &- S\epsilon'(c_{11}x_1 + \ldots) S x_1 \epsilon - S\epsilon'(c_{12} x_1 + \ldots) S x_2 \epsilon - S\epsilon'(c_{13} x_1 + \ldots) S x_3 \epsilon \\ &- S\epsilon(c_{11}' x_1' + \ldots) S x_1' \epsilon' - S\epsilon(c_{12}' x_1' + \ldots) S x_2' \epsilon' - S\epsilon(c_{13}' x_1' + \ldots) S x_3' \epsilon' \\ &+ S(c_{11} x_1 + \ldots)(c_{11}' x_1' + \ldots) S x_1 \epsilon S x_1' \epsilon' + \text{five other terms.} \end{aligned}$$

Now

$$ES\epsilon'(c_{11}x_1 + c_{12}x_2 + c_{13}x_3) S x_1 \epsilon = \kappa S x_1 (c_{11} x_1 + \ldots) = \kappa (c_{11} S x_1^2 + c_{12} S x_1 x_2 + c_{13} S x_1 x_3) = \kappa,$$

and similarly for the other five like terms.

Also $ESx_1\epsilon \; Sx_1'\epsilon' = \kappa \; Sx_1 x_1'$, so that the last six terms, after rearrangement, reduce to

$$\kappa T = \kappa \Sigma c_{rs} c_{tu}' S x_r x_t' S x_s x_u',$$

with summation over all r, s, t, u from 1 to 3.

Thus

$$ESee' = (n - 6 + T)\kappa,$$

or, if k constants are fitted,

$$ESee' = (n - 2k + T)\kappa.$$

If there is complete correlation between x_1 and x_1', x_2 and x_2', etc., it is easily shown that T has the value k, confirming the results obtained above. If there is no correlation then T equals zero. In general T must lie between k and zero, as will now be shown.

Each set of independent variates may be replaced by an orthogonal set, having unit sum of squares. Let these be $\xi_1, \xi_2, \ldots,$ and $\xi_1', \xi_2', \ldots$ Then

$$S\xi_1^2 = S\xi_2^2 = \ldots = S\xi_1'^2 = S\xi_2'^2 = \ldots = 1, \text{ and } S\xi_1\xi_2 = \ldots = S\xi_1'\xi_2' = \ldots = 0.$$

We also have, for the transformed variates,

$$c_{11} = c_{22} = \ldots = c_{11}' = c_{22}' = \ldots = 1, \quad \text{and} \quad c_{12} = \ldots = c_{12}' = \ldots = 0.$$

Hence

$$\begin{aligned} T = {} & (S\xi_1\xi_1')^2 + (S\xi_1\xi_2')^2 + (S\xi_1\xi_3')^2 + \ldots \\ & + (S\xi_2\xi_1')^2 + (S\xi_2\xi_2')^2 + (S\xi_2\xi_3')^2 + \ldots \\ & + \ldots\ldots\ldots \end{aligned}$$

If we take the regression of ξ_1 on $\xi_1', \xi_2', \xi_3', \ldots$ the sum of squares accounted for by this regression will be given by the terms in the first line of the above expression for T. This sum of squares cannot exceed $S(\xi_1^2)$, *i.e.* unity. Hence T cannot exceed k, and is clearly always positive.

Thus instead of dividing the sum of products of the residuals by the residual degrees of freedom, $n - k$, we must divide by some number, $n - 2k + T$, lying between $n - 2k$ and $n - k$. If the d and d' matrices have been evaluated the exact value of T can be easily computed, being equal to the sum of the products of the like terms in the two matrices. Symbolically

$$T = \Sigma d_{rs} d_{rs}'$$

with summation over all r and s from 1 to k.

Effect of Errors in the Independent Variates.

It is well known that random errors affecting the dependent variate do not introduce any constant errors into the regression coefficients, but merely decrease their accuracy. It is also sometimes stated that the independent variates, even if subject to error, "can be regarded as errorless." This, however, is only true in certain circumstances. Errors in

the independent variates do not affect the validity of the ordinary tests of significance, provided that only differences from zero are being tested, and the coefficients as ordinarily evaluated will always be the best coefficients for the purpose of prediction from subsequent observations subject to similar errors. On the other hand errors in the independent variates do introduce certain biases into the coefficients, their absolute values being in general reduced, and these biases may be of importance if the theoretical values of the coefficients are under consideration.

Thus, for example, if we have reason for believing that a regression coefficient has a theoretical value $(\beta_1)_0$, it is not correct, if the x's are subject to error, to test the difference $b_1 - (\beta_1)_0$ to see if the observations conform with the theory, except in the case when the expected theoretical values of all the coefficients are zero. Similarly, when testing whether the observed difference between two coefficients determined from different sets of observations indicates a real effect, account must be taken of possible errors in the independent variates. In this case, however, both the coefficients will be similarly reduced if the errors of the independent variates and their sums of squares and products are similar in both sets of observations.

The actual biases cannot be estimated unless the variances and covariances of the errors affecting the independent variates are known. Even if they are known the general problem of tests of significance is of some complexity, and will not be further discussed here. The possibility of disturbances of this nature should not, however, be lost sight of when the interpretation of the results of an ordinary regression analysis is undertaken.

Simplification when only One of the Independent Variates has Different Values.

The formulæ for the case in which some of the pairs of independent variates are identical can be easily deduced from those given above. When only one pair of variates, say x_1 and x_1', are different the d and d' matrices reduce to

$$\begin{matrix} d_{11} & 0 & 0 \\ d_{21} & 1 & 0 \\ d_{31} & 0 & 1 \end{matrix} \quad \text{and} \quad \begin{matrix} d'_{11} & d'_{12} & d'_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

and the last two terms in the formula for the sum of the products of the residuals cancel out with $b_2 Sx_2 y'$ and $b_3 Sx_3 y'$ respectively.

EXAMPLE.

The regression of percentage nitrogen in the barley grain (y) on the yield (x_1), sowing date (x_2), and April–June rainfall (x_3), was calculated for several of the Hoosfield plots at Rothamsted (Yates and Watson, 1939). A series of thirty-four years' data was available. To eliminate the effects of slow changes in yield the deviations from six- and seven-year means were taken.

(It may be noted, in passing, that the yield may itself be influenced to a greater or less extent by sowing date and rainfall, and consequently the regressions represent the residual and not the total effects of these factors. To estimate the total effects it would be necessary to exclude the yields from the regressions. Actually the yields show little correlation with the other factors, and consequently the two sets of coefficients differ but little. Moreover, it would be wrong to consider the regression coefficients on yield as representing the "effects" of the changes in yield on the nitrogen percentages. Both variates are end products of a complex of causes too complicated to determine fully. Their interrelations may, nevertheless, be of considerable interest.)

The regression coefficients obtained, and their standard errors, are shown in Table I. It is necessary to determine which of the observed differences in the regression coefficients are significant.

TABLE I.—HOOSFIELD BARLEY.

Regression of Percentage Nitrogen in the Grain on Yield, April–June Rainfall and Sowing Date, 1893–1911, 1914–1928. (Mean Rainfall: 6·01 inches. Mean Sowing Date: 21st March.)

Plot.	Mean yield, cwt.	Mean N %.	Partial Regressions of N % on			Residual S.E.	Variance accounted for.
			Yield (cwt).	Rain (inches).	Sowing Date (days).		
1 − 0	5·0	1·474	−·0255* ± ·0114	−·0265 ± ·0149	+·00517** ± ·00179	·138	37 %
1 − *A*	9·0	1·676	−·0290** ± ·0100	−·0477* ± ·0178	+·00257 ± ·00214	·166	40 %
4 − 0	7·5	1·510	−·0246** ± ·0088	−·0304 ± ·0157	+·00494** ± ·00189	·146	41 %
4 − *A*	17·0	1·544	−·0113 ± ·0058	−·0393** ± ·0144	+·00394* ± ·00176	·134	43 %
7 − 2	18·3	1·848	−·0006 ± ·0048	−·0002 ± ·0148	+·00484* ± ·00177	·137	16 %

In view of the somewhat laborious calculations required, the differences to be tested should be chosen with care. The main apparent differences are those between the regression coefficients for yield and rainfall on the dunged plot ($7-2$) and the other plots, and between the coefficients for yield on the plot with complete artificials ($4-A$) and plots $1-0$ (no

manure), $1-A$ (nitrogen only), and $4-0$ (minerals only). It was therefore decided to test plot $7-2$ against plot $4-A$, and plot $4-A$ against plot $1-A$ (which showed the highest regression on yield). Details of the computations for the latter test are reproduced here.

Table II gives the matrices of sums of squares and products of the original variates, the c matrices, and the values of the regression coefficients, in the original units (adjusted by suitable powers of 10). The whole of this table is obtained in the course of the ordinary regression analyses.

TABLE II.—SUMS OF SQUARES AND PRODUCTS, ETC., FOR PLOTS $1-A$ AND $4-A$.

	x_1	x_2	x_3	y	c_{1r}	c_{2r}	c_{3r}	b
x_1	+3·5199	+·1703	−·0540	−1·0071	+·2866	−·0494	+·0052	−·2591
x_2		+·9617	−·2443	−·5654		+1·1559	+·4224	−·4768
x_3			+·6622	+·3005			+1·6663	+·2567
y				+1·3239				
	x_1'	x_2'	x_3'	y'	c_{1r}'	c_{2r}'	c_{3r}'	b'
x_1'	+6·8538	+·2549	−·4260	−·9617	+·1522	−·0171	−·0916	−·1013
x_2'		+·9617	−·2443	−·4996		+1·1492	+·4130	−·3927
x_3'			+·6622	+·3997			+1·7214	+·3935
y'				+·9188				

Table III gives the sums of cross products of the two sets of variates. Since $x_2 = x_2'$ and $x_3 = x_3'$ the bracketed figures do not require computation, being derived from Table II.

TABLE III.—SUMS OF CROSS PRODUCTS

	x_1'	x_2'	x_3'	y'
x_1	+3·0925	(+·1703)	(−·0540)	−·7433
x_2	(+·2549)	(+·9617)	(−·2443)	(−·4996)
x_3	(−·4260)	(−·2443)	(+·6622)	(+·3997)
y	−1·1296	(−·5654)	(+·3005)	+·8701

Table IV gives the d and d' matrices, and Table V the C matrix.

TABLE IV.—THE d AND d' MATRICES

+·8715	0	0		+·4628	+·1205	+·2607
−·0381	1	0		0	1	0
−·5861	0	1		0	0	1

TABLE V.—THE C MATRIX

+·1326	−·0149	+·0799
−·0229	+1·1499	+·4095
+·0024	+·4230	+1·6677

Table VI shows the analyses of variance of y and y'. These also form part of the ordinary analyses.

TABLE VI.—ANALYSES OF VARIANCE OF y AND y'

		y^2		y'^2	
	D.F.	S.S.	M.S.	S.S.	M.S.
Regression . .	3	·6077	·20257	·4508	·15027
Residuals . .	26	·7162	·02754	·4680	·01800
Total .	29	1·3239	·04565	·9188	·03168

Table VII shows the calculation of the residual sum of products. The value of T, obtained from the d and d' matrices, is

$$T = +\cdot 8715 \times \cdot 4628 + 1 + 1 = 2\cdot 4033.$$

TABLE VII.—CALCULATION OF RESIDUAL SUM OF PRODUCTS

Syy'	$+\cdot 8701$	Syy'	$+\cdot 8701$
$-(b_1Sx_1y' + b_2Sx_2y' + b_3Sx_3y')$	$-\cdot 5334$	$-(b_1'Sx_1'y + b_2'Sx_2'y + b_3'Sx_3'y)$	$-\cdot 4547$
$-b_1'(Sx_1'y - b_1Sx_1x_1' - \dots)$	$-\cdot 0099$	$-b_1(Sx_1y' - b_1'Sx_1x_1' - \dots)$	$-\cdot 0886$
	$+\cdot 3268$		$+\cdot 3268$

Hence the divisor is $29 - 6 + 2\cdot 4033 = 25\cdot 4033$, giving a residual covariance of $+\cdot 01286$.

The estimated variance of $b_1 - b_1'$ is therefore

$$+\cdot 2866 \times \cdot 02754 + \cdot 1522 \times \cdot 01800 - 2 \times \cdot 1326 \times \cdot 01286 = \cdot 007222.$$

Hence the standard error of $b_1 - b_1'$ is $\pm \cdot 0850$. The actual difference is $-\cdot 1578$, which is less than twice the estimated standard error. This is the largest of the differences between the four plots not receiving organic manure. There is therefore no evidence of any real difference between these coefficients.

The standard errors of $b_2 - b_2'$ and $b_3 - b_3'$ may be computed in a similar manner. We obtain

$$b_2 - b_2' = -\cdot 0841 \pm \cdot 151,$$
$$b_3 - b_3' = -\cdot 1368 \pm \cdot 184.$$

There is thus no evidence of any real difference in either of these coefficients.

The results of the other test performed on the data of Table I, namely

between the coefficients of plots $4-A$ and $7-2$, may also be given for comparison. They are

$$b_1 - b_1' = -\cdot 0964 \pm \cdot 055,$$
$$b_2 - b_2' = -\cdot 3908 \pm \cdot 128,$$
$$b_3 - b_3' = -\cdot 0908 \pm \cdot 156.$$

The difference in the regressions on rainfall is clearly significant. The difference in the regressions on yield does not quite reach significance, but in view of the fact that we are here testing the smallest of four similar differences, we may conclude that the regression on yield is significantly lower on the organic than on the average of the inorganic plots.

SUMMARY.

Tests of significance of the differences of regression coefficients derived from two sets of correlated dependent and independent variates are described. The necessary computations are reduced to a systematic and easily calculable form, and are illustrated by a numerical example.

REFERENCES TO LITERATURE.

COCHRAN, W. G., 1935. "A Note on the Influence on the Yield of Cereals in Relation to Manurial Treatment," *Journ. Agric. Sci.*, vol. xxv, pp. 510–522.

FISHER, R. A., 1924. "The Influence of Rainfall on the Yield of Wheat at Rothamsted," *Phil. Trans. Roy. Soc. London*, B, vol. ccxiii, pp. 89–142.

YATES, F., and WATSON, D. J., 1939. "Factors influencing the Percentage of Nitrogen in the Barley Grain of Hoosfield," *Journ. Agric. Sci.*, vol. xxix (in the press).

(*Issued separately August* 10, 1939.)

XIX.—**On the Reciprocation of Certain Matrices.** By **A. R. Collar,** B.A., B.Sc., Aerodynamics Department, the National Physical Laboratory.* *Communicated by* Dr A. C. Aitken, F.R.S.

(MS. received April 18, 1939. Read July 3, 1939.)

§ 1. *Introductory.*—In a recent paper (Frazer, Jones, and Skan, 1937) some methods are discussed for the approximate representation of functions by means of polynomials. The coefficients in the polynomials are determined by the equation

$$c = M^{-1}h,$$

where c is the column of coefficients, h is a column of known constants, and M is a matrix which depends on the method of representation adopted. The present paper shows how the reciprocal matrix M^{-1} can be computed rapidly and simply in the two cases where M is a moment matrix or an alternant matrix.

Equation (1), which provides an elegant method for the computation of the reciprocal of a moment matrix, is due to Dr A. C. Aitken. The author is grateful to Dr Aitken for permission to describe the method, and for many other helpful suggestions.

§ 2. *The Reciprocal of a Moment Matrix.*— Suppose the (i, j)th element of the persymmetric moment matrix

$$M = \begin{bmatrix} m_1 & m_2 & m_3 & \cdot & \cdot & \cdot & m_n \\ m_2 & m_3 & m_4 & \cdot & \cdot & \cdot & m_{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ m_n & m_{n+1} & m_{n+2} & \cdot & \cdot & \cdot & m_{2n-1} \end{bmatrix}$$

to be given by

$$m_{i+j-1} = \int_a^b w(x)x^{i+j-2}dx,$$

where $w(x)$ is some given weight function. Also, suppose there exists a set of polynomials

$$p_i(x) = p_{i1} + p_{i2}x + \dots + p_{ii}x^{i-1}$$

* The author is indebted to the Aeronautical Research Committee for permission to publish this paper.

which satisfy the orthogonal relations

$$\left.\begin{aligned}\int_a^b w(x)p_i(x)p_j(x)dx &= 0, \qquad i \neq j,\\ &= d_{ii}, \qquad i = j.\end{aligned}\right\}$$

Let

$$P = \begin{bmatrix} p_{11} & 0 & 0 & . & . & . & 0 \\ p_{21} & p_{22} & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ p_{n1} & p_{n2} & p_{n3} & . & . & . & p_{nn} \end{bmatrix}.$$

Then the (i, j)th element of the product PMP' is

$$[p_{i1}, p_{i2}, \ldots, p_{ii}, 0, \ldots, 0]M\{p_{j1}, \ldots, p_{jj}, 0, \ldots, 0\}$$

$$= \sum_{r=1}^{i}\sum_{s=1}^{j} p_{ir}p_{js}m_{r+s-1}$$

$$= \sum_{r=1}^{i}\sum_{s=1}^{j} p_{ir}p_{js}\int_a^b w(x)x^{r+s-2}dx$$

$$= \int_a^b w(x)p_i(x)p_j(x)dx.$$

In view of the orthogonal properties of the polynomials, therefore, the (i, j)th element of the product vanishes except for $i=j$. Hence

$$PMP' = D,$$

where D is the diagonal matrix of which the (i, i)th element is d_{ii}. On reciprocation,

$$M^{-1} = P'D^{-1}P. \qquad (1)$$

The formula (1) provides a simple method of computing the reciprocals of moment matrices which are derived from weight functions for which the orthogonal polynomials $p_i(x)$ exist and are known. For example, if $w(x) = 1$ and the range of integration is (0, 1) or (−1, 1), the appropriate Legendre polynomials are used. If $w(x) = x^p(1-x)^q$ and (a, b) is (0, 1), the Jacobi polynomials are used. Similarly, Laguerre, Sonine, Hermite, or Tchebychef polynomials are used for their appropriate weight functions and ranges of integration. (See, for instance, Courant and Hilbert, 1924. The use of Legendre and Tchebychef polynomials is exemplified in a paper by Aitken, 1933, which also contains a short bibliography.)

If in a given case several or all of the reciprocal matrices M^{-1} from order 1 to n are required, they can be built up successively without the

use of equation (1) in each instance. Let the matrices in (1) be of order r, and let them be symmetrically partitioned, so that (1) may be written:

$$\mathrm{M}^{-1}=\begin{bmatrix}\alpha' & \beta' \\ 0 & \gamma'\end{bmatrix}\begin{bmatrix}\delta_1 & 0 \\ 0 & \delta_2\end{bmatrix}\begin{bmatrix}\alpha & 0 \\ \beta & \gamma\end{bmatrix},$$

where α, δ_1 are of order s, and γ, δ_2 of order $r-s$. By expansion,

$$\begin{aligned}\mathrm{M}^{-1}&=\begin{bmatrix}\alpha'\delta_1\alpha+\beta'\delta_2\beta, & \beta'\delta_2\gamma \\ \gamma'\delta_2\beta, & \gamma'\delta_2\gamma\end{bmatrix}\\ &=\begin{bmatrix}\alpha'\delta_1\alpha & 0 \\ 0 & 0\end{bmatrix}+\begin{bmatrix}\beta' \\ \gamma'\end{bmatrix}\delta_2[\beta \quad \gamma]. \qquad \qquad (2)\end{aligned}$$

The submatrix $\alpha'\delta_1\alpha$ is the reciprocal matrix M^{-1} of order s. When this is known, the matrix M^{-1} of order r is obtained from it by bordering it appropriately with ciphers and adding it to the second term in (2). In particular, when $r-s=1$, the matrix $[\beta \quad \gamma]$ is simply the row of coefficients in $p_r(x)$ and δ_2 is the scalar d_{rr}^{-1}. The second term in (2) is then a matrix of unit rank, all the rows and all the columns being proportional to the array of coefficients in $p_r(x)$; it is therefore very easily computed. The successive matrices M^{-1} of increasing order are thus very simply obtained.

It is of interest to note here an analogy with a method described by Frazer, Duncan, and Collar (1938) for the reciprocation of any numerical matrix: in this method also the reciprocal is obtained by the addition of a matrix of unit rank to the reciprocal of the leading first minor bordered with ciphers.

§ 3. *An Example.*—Consider the case where the weight function $w(x)$ is unity and the range of integration is (0, 1). Then

$$\mathrm{M}=\begin{bmatrix}1 & \frac{1}{2} & \frac{1}{3} & \cdot & \cdot & \cdot & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdot & \cdot & \cdot & \frac{1}{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdot & \cdot & \cdot & \frac{1}{2n-1}\end{bmatrix}.$$

The Legendre polynomials required in this instance are

$$l_i(x)=\frac{1}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}x^{i-1}(1-x)^{i-1},$$

the coefficient of x^{r-1} being

$$l_{ir}=\frac{(-1)^{r-1}(i+r-2)!}{(r-1)!\,(r-1)!\,(i-r)!}.$$

The first four polynomials are

$$\begin{aligned} l_1(x) &= 1, \\ l_2(x) &= 1 - 2x, \\ l_3(x) &= 1 - 6x + 6x^2, \\ l_4(x) &= 1 - 12x + 30x^2 - 20x^3. \end{aligned}$$

Now

$$\left.\begin{aligned} \int_0^1 l_i(x) l_j(x)\,dx &= 0, & i \neq j, \\ &= \frac{1}{2i-1}, & i = j. \end{aligned}\right\}$$

Hence

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & . & . & . \\ 1 & -2 & 0 & 0 & . & . & . \\ 1 & -6 & 6 & 0 & . & . & . \\ 1 & -12 & 30 & -20 & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}; \quad D^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & . & . & . \\ 0 & 3 & 0 & 0 & . & . & . \\ 0 & 0 & 5 & 0 & . & . & . \\ 0 & 0 & 0 & 7 & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}.$$

Thus for the third-order matrix

$$M^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -6 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 1 & -6 & 6 \end{bmatrix} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}.$$

To obtain the reciprocal matrix of the fourth order, equation (2) may be applied; it becomes

$$\begin{bmatrix} 9 & -36 & 30 & 0 \\ -36 & 192 & -180 & 0 \\ 30 & -180 & 180 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + 7 \begin{bmatrix} 1 & -12 & 30 & -20 \\ -12 & 144 & -360 & 240 \\ 30 & -360 & 900 & -600 \\ -20 & 240 & -600 & 400 \end{bmatrix} = \begin{bmatrix} 16 & -120 & 240 & -140 \\ -120 & 1200 & -2700 & 1680 \\ 240 & -2700 & 6480 & -4200 \\ -140 & 1680 & -4200 & 2800 \end{bmatrix}.$$

When $w(x) = 1$ and the range of integration is $(0, 1)$ there is an even simpler form for the reciprocal matrix M^{-1}. If the order is n, the typical element in M^{-1} is evidently $\sum_{r=1}^{n} (2r-1) l_{ri} l_{rj}$. Now

$$l_{ri} = \frac{(-1)^{i-1}(r+i-2)!}{(i-1)!\,(i-1)!\,(r-i)!}$$

and

$$l_{r+1,\,i} = \frac{(-1)^{i-1}(r+i-1)!}{(i-1)!\,(i-1)!\,(r-i+1)!} = \frac{(r+i-1)}{(r-i+1)} l_{ri}.$$

Hence if

$$f(r) = \frac{(r-i)(r-j)}{(i+j-1)} l_{ri} l_{rj},$$

then

$$f(r+1)-f(r)=\frac{(r+i-1)(r+j-1)-(r-i)(r-j)}{(i+j-1)}l_{ri}l_{rj}$$

$$=(2r-1)l_{ri}l_{rj}.$$

Hence

$$\sum_{r=1}^{n}(2r-1)l_{ri}l_{rj}=f(n+1),$$

since $f(1)=0$. The typical term in M^{-1} is thus

$$f(n+1)=\frac{(n-i+1)(n-j+1)}{(i+j-1)}l_{n+1,\,i}\,l_{n+1,\,j},$$

a formula which involves no summation. If Δ is the diagonal matrix of which the $(i,\ i)$th element is $(n-i+1)l_{n+1,\,i}$, then the preceding results are equivalent to

$$M^{-1}=\Delta M\Delta. \qquad . \qquad . \qquad . \qquad . \qquad . \qquad . \qquad (3)$$

As a numerical example let $n=4$. Then the numbers in the diagonal matrix Δ are given by

$$(5-i)l_{5i}=\frac{(-1)^{i-1}(3+i)!}{(i-1)!\,(i-1)!\,(4-i)!}.$$

Hence

$$M^{-1}=\begin{bmatrix}4&0&0&0\\0&-60&0&0\\0&0&180&0\\0&0&0&-140\end{bmatrix}\begin{bmatrix}1&\frac{1}{2}&\frac{1}{3}&\frac{1}{4}\\\frac{1}{2}&\frac{1}{3}&\frac{1}{4}&\frac{1}{5}\\\frac{1}{3}&\frac{1}{4}&\frac{1}{5}&\frac{1}{6}\\\frac{1}{4}&\frac{1}{5}&\frac{1}{6}&\frac{1}{7}\end{bmatrix}\begin{bmatrix}4&0&0&0\\0&-60&0&0\\0&0&180&0\\0&0&0&-140\end{bmatrix}$$

$$=\begin{bmatrix}16&-120&240&-140\\-120&1200&-2700&1680\\240&-2700&6480&-4200\\-140&1680&-4200&2800\end{bmatrix},$$

in agreement with the previous result.

§ 4. *A Second Example.*—Suppose $w(x)=x^p$, the range of integration being $(0,\ 1)$. In this case the functions to be used are the Jacobi polynomials

$$h_i(x)=\frac{x^{-p}}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}x^{p+i-1}(1-x)^{i-1},$$

for which

$$\left.\begin{aligned}\int_0^1 x^p h_i(x) h_j(x) &= 0, & i \neq j,\\ &= \frac{1}{p+2i-1}, & i = j.\end{aligned}\right\}$$

The example of § 3 is thus a particular case of this example, and the moment matrices defined by different values of p are all submatrices (not necessarily symmetrically placed) of the moment matrix of § 3. Thus

$$M = \begin{bmatrix} \frac{1}{p+1} & \frac{1}{p+2} & \frac{1}{p+3} & \cdots \\ \frac{1}{p+2} & \frac{1}{p+3} & \frac{1}{p+4} & \cdots \\ \frac{1}{p+3} & \frac{1}{p+4} & \frac{1}{p+5} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

The typical coefficient in $h_i(x)$ is

$$h_{ir} = \frac{(-1)^{r-1}(p+i+r-2)!}{(p+r-1)!\,(r-1)!\,(i-r)!},$$

and the generalisations of the formulæ of § 3 are therefore

$$P = \begin{bmatrix} 1, & 0, & 0, & \cdots \\ (p+1), & -(p+2), & 0, & \cdots \\ \frac{(p+1)(p+2)}{2}, & -(p+2)(p+3), & \frac{(p+3)(p+4)}{2}, & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}; \quad D^{-1} = \begin{bmatrix} (p+1), & 0, & 0, & \cdots \\ 0, & (p+3), & 0, & \cdots \\ 0, & 0, & (p+5), & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

As for the case considered in § 3, there is here a simpler form for M^{-1}. The (i, j)th element of M^{-1} is $\sum_{r=1}^{n}(p+2r-1)h_{ri}h_{rj}$, and this summation is equal to $f(n+1, p)$, where

$$f(r, p) = \frac{(r-i)(r-j)}{(p+i+j-1)} h_{ri} h_{rj}.$$

Hence if Δ is now the diagonal matrix of which the (i, i)th element is $(n-i+1)h_{n+1, i}$, then as before

$$M^{-1} = \Delta M \Delta.$$

Suppose for example that $p=1$ and $n=4$. Then

$$M=\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \end{bmatrix},$$

and

$$\Delta=\begin{bmatrix} 20 & 0 & 0 & 0 \\ 0 & -180 & 0 & 0 \\ 0 & 0 & 420 & 0 \\ 0 & 0 & 0 & -280 \end{bmatrix},$$

so that

$$M^{-1}=\Delta M\Delta=\begin{bmatrix} 200 & -1200 & 2100 & -1120 \\ -1200 & 8100 & -15120 & 8400 \\ 2100 & -15120 & 29400 & -16800 \\ -1120 & 8400 & -16800 & 9800 \end{bmatrix}.$$

Since $(\Delta M)^2=I$, ΔM is a square root of the unit matrix. It can be shown that the latent roots of ΔM are $+1$, -1, in alternation: these signs appear in the principal diagonal of Δ also, so that the determinant of M is positive and is given by

$$\begin{vmatrix} \frac{1}{p+1}, & \frac{1}{p+2}, & \ldots, & \frac{1}{p+n} \\ \frac{1}{p+2}, & \frac{1}{p+3}, & \ldots, & \frac{1}{p+n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{p+n}, & \frac{1}{p+n+1}, & \ldots, & \frac{1}{p+2n-1} \end{vmatrix} = \frac{p!\,(p+1)!\,\ldots\,(p+n-1)!\,[0!\,1!\,\ldots\,(n-1)!]^2}{(p+n)!\,(p+n+1)!\,\ldots\,(p+2n-1)!}.$$

§ 5. *The Reciprocal of an Alternant Matrix.*—Consider the simple alternant matrix

$$M=\begin{bmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_n \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{n-1} & x_2^{n-1} & \ldots & x_n^{n-1} \end{bmatrix}.$$

Formally, the reciprocal of this matrix is readily written down. Let

$$\begin{aligned} f_i(x) &= (x-x_1)(x-x_2)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n) \\ &= p_{in}x^{n-1}+p_{i,n-1}x^{n-2}+\ldots+p_{i1} \\ &= [p_{i1},\ p_{i2}\ldots p_{in}]\{1,\ x\ldots,\ x^{n-1}\}, \end{aligned}$$

where $p_{in} = 1$. Then evidently

$$[p_{i1}, \ p_{i2} \ldots p_{in}]\{1, \ x_j \ldots x_j^{n-1}\} = f_i(x_j) \\ \left.\begin{aligned} &= \ 0, \quad i \neq j, \\ &= f_i(x_i), \ i = j. \end{aligned}\right\}$$

It follows that the ith row of M_r^{-1} is

$$\frac{1}{f_i(x_i)}[p_{i1}, \ p_{i2} \ldots p_{in}],$$

or

$$M^{-1} = \begin{bmatrix} \frac{1}{f_1(x_1)} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \frac{1}{f_2(x_2)} & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \frac{1}{f_n(x_n)} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & \cdot & \cdot & \cdot & p_{1n} \\ p_{21} & p_{22} & \cdot & \cdot & \cdot & p_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{n1} & p_{n2} & \cdot & \cdot & \cdot & p_{nn} \end{bmatrix} \tag{4}$$

In practice, the method of construction would be to form the polynomial $F(x) = \prod_{r=1}^{n} (x - x_r)$ and to remove the factors $x - x_1$, $x - x_2$, . . . in turn from $F(x)$ to obtain the polynomials $f_i(x)$. The numbers $f_i(x_i)$ would be obtained either by substitution of x_i for x in $f_i(x)$, or by evaluation of the appropriate products of differences $x_i - x_j$.

It is also possible to express the matrix on the right of (4) as a product involving only M and the coefficients of $F(x)$: the construction of the polynomials $f_i(x)$ is thus avoided, since the numbers $f_i(x_i)$ may be obtained either as $F^{(1)}(x_i)$, where $F^{(1)}(x)$ is the differential coefficient of $F(x)$, or, as before, as difference products. This form is particularly advantageous when a column $M^{-1}h$ is required, as indicated in § 1, since in this case the explicit evaluation of M^{-1} is not necessary.

Consider the product R′PS, where

$$R = \{1, \ x_r, \ \ldots, \ x_r^{n-1}\}, \quad S = \{1, \ x_s, \ \ldots, \ x_s^{n-1}\},$$

and P is the persymmetric matrix

$$P = \begin{bmatrix} P_{n-1} & P_{n-2} & \cdot & \cdot & \cdot & P_1 & 1 \\ P_{n-2} & P_{n-3} & \cdot & \cdot & \cdot & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \end{bmatrix},$$

the non-zero elements of which are the coefficients in

$$F(x) = \prod_{i=1}^{n} (x - x_i) = x^n + P_1 x^{n-1} + \ . \ . \ . \ + P_{n-1} x + P_n.$$

On expansion,

$$R'PS = \frac{F(x_r) - F(x_s)}{(x_r - x_s)} \qquad . \quad . \quad . \quad . \quad . \qquad (5)$$

$$\left.\begin{aligned} &= 0, && r \neq s, \\ &= F^{(1)}(x_r), && r = s. \end{aligned}\right\}$$

If

$$\Phi = \begin{bmatrix} F^{(1)}(x_1) & 0 & . & . & . & 0 \\ 0 & F^{(1)}(x_2) & . & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & F^{(1)}(x_n) \end{bmatrix},$$

then equation (5) shows that $M'PM = \Phi$, and on reciprocation

$$M^{-1} = \Phi^{-1}M'P. \qquad . \quad . \quad . \quad . \quad . \qquad (6)$$

Since Φ^{-1} is the matrix on the left in (4), the matrix on the right is equal to $M'P$.

Suppose, for example, that $n = 3$ and $x_1 = 2$, $x_2 = 3$, $x_3 = 5$. Then

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 9 & 25 \end{bmatrix}.$$

To illustrate (4), the product

$$F(x) = (x-2)(x-3)(x-5) = x^3 - 10x^2 + 31x - 30$$

is evaluated, and extraction of the factors one at a time gives

$$\begin{aligned} f_1(x) &= x^2 - 8x + 15, \\ f_2(x) &= x^2 - 7x + 10, \\ f_3(x) &= x^2 - 5x + 6, \end{aligned}$$

so that $f_1(2) = 3$, $f_2(3) = -2$, $f_3(5) = 6$. Hence

$$M^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 15 & -8 & 1 \\ 10 & -7 & 1 \\ 6 & -5 & 1 \end{bmatrix}.$$

For formula (6), $F(x)$ is required and

$$F^{(1)}(x) = 3x^2 - 20x + 31,$$

giving $F^{(1)}(2) = 3$, $F^{(1)}(3) = -2$, $F^{(1)}(5) = 6$. Then

$$M^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \end{bmatrix} \begin{bmatrix} 31 & -10 & 1 \\ -10 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

It will be seen that, in the application of the formulæ (4) and (6) to the evaluation of a column $M^{-1}h$, the process of forming the polynomials

$f_i(x)$ in (4) is replaced in (6) by an additional multiplication of a square matrix and a column; this additional multiplication, namely Ph, is, however, relatively simple in view of the triangular nature of P.

§ 6. *The Case of a Confluent Alternant Matrix.*—When two or more of the parameters x_i in an alternant matrix coalesce, the matrix becomes singular, and it is then usually replaced by a confluent alternant matrix. Suppose, for instance, a of the quantities x_i become equal to x_r; then the a columns

$$R=\{1, \quad x_r, \quad \ldots, \quad x_r^{n-1}\}$$

are replaced by

$$R_p=\frac{1}{p!}\frac{d^pR}{dx_r^p} \quad (p=0, 1, \ldots, a-1).$$

Similarly if b of the parameters become equal to x_s, the corresponding columns are

$$S_q=\frac{1}{q!}\frac{d^qS}{dx_s^q} \quad (q=0, 1, \ldots, b-1).$$

A typical confluent alternant matrix is

$$M=\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ x_1 & 1 & x_3 & 1 & 0 \\ x_1^2 & 2x_1 & x_3^2 & 2x_3 & 1 \\ x_1^3 & 3x_1^2 & x_3^3 & 3x_3^2 & 3x_3 \\ x_1^4 & 4x_1^3 & x_3^4 & 4x_3^3 & 6x_3^2 \end{bmatrix}. \qquad (7)$$

To obtain the generalisation of equation (6) it is evidently necessary to evaluate products of the types $R_p'PS_q$ and $R_p'PR_q$. Direct differentiation of equation (5) with respect to x_r and x_s yields a result which may be written

$$R_p'PS_q=\sum_{i=0}^{p}\frac{(-1)^{p-i}(p+q-i)!\,F^{(i)}(x_r)}{(p-i)!\,i!\,q!\,(x_r-x_s)^{p+q-i+1}}+\sum_{i=0}^{q}\frac{(-1)^{q-i}(q+p-i)!\,F^{(i)}(x_s)}{(q-i)!\,i!\,p!\,(x_s-x_r)^{q+p-i+1}},$$

where $F^{(i)}(x)$ is the ith differential coefficient of $F(x)$. But since $F(x)$ contains the factor $(x-x_r)^a$, $F^{(i)}(x_r)=0$ for $i=0, 1, \ldots, p, \ldots, a-1$; similarly $F^{(i)}(x_s)=0$ for $i=0, 1, \ldots, q, \ldots, b-1$. Hence

$$R_p'PS_q=0. \qquad (8)$$

Next, if in equation (5) $F(x_r)$ is written as $F(x_s+x_r-x_s)$ and expanded in a Taylor's series, the result is

$$R'PS=\sum_{i=1}^{n}\frac{(x_r-x_s)^{i-1}}{i!}F^{(i)}(x_s),$$

and differentiation of this equation with respect to x_r and x_s yields

$$R_p'PS_q = \sum_{i=1}^{n-p-q} \frac{(p+i-1)!}{p!\,(p+q+i)!} \frac{(x_r-x_s)^{i-1}}{(i-1)!} F^{(p+q+i)}(x_s).$$

When $x_r = x_s$, this gives

$$R_p'PR_q = \frac{1}{(p+q+1)!} F^{(p+q+1)}(x_r). \qquad . \quad . \quad . \quad (9)$$

It follows from (8) and (9) that the product $M'PM$ is a diagonal matrix of submatrices. The order of each submatrix is the number of columns in which the corresponding quantity x_i occurs: if x_r occurs in a columns, the corresponding submatrix is of order a. Moreover, since $F^{(i)}(x_r) = 0$ for $i = 0, 1, \ldots, a-1$, the submatrix has the special persymmetric form

$$\begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 & \phi_a \\ 0 & 0 & \cdot & \cdot & \cdot & \phi_a & \phi_{a+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \phi_a & \phi_{a+1} & \cdot & \cdot & \cdot & \phi_{2a-2} & \phi_{2a-1} \end{bmatrix},$$

where $\phi_i = \frac{1}{i!} F^{(i)}(x_r)$. This particular type of submatrix is easily reciprocated numerically; the reciprocal is also persymmetric and may be written

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\,\gamma & \beta & \alpha \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\,\beta & \alpha & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha & 0\,\cdot & \cdot & \cdot & \cdot & \cdot\,0 & 0 & 0 \end{bmatrix},$$

the elements being determined by the conditions

$$\alpha\phi_a = 1, \qquad \beta\phi_a + \alpha\phi_{a+1} = 0, \qquad \gamma\phi_a + \beta\phi_{a+1} + \alpha\phi_{a+2} = 0,$$

and so on. Accordingly, the reciprocal of the diagonal matrix of submatrices obtained from $M'PM$ is easily evaluated, so that equation (6) may be used to calculate the reciprocals of confluent as well as simple alternant matrices.

As a numerical example, let $x_1 = 2$ and $x_2 = 3$ in (7), so that

$$M = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 3 & 1 & 0 \\ 4 & 4 & 9 & 6 & 1 \\ 8 & 12 & 27 & 27 & 9 \\ 16 & 32 & 81 & 108 & 54 \end{bmatrix}.$$

Then

$$F(x) = (x-2)^2(x-3)^3 = x^5 - 13x^4 + 67x^3 - 171x^2 + 216x - 108,$$

so that

$$\frac{1}{1!}F^{(1)}(2)=0, \qquad \frac{1}{2!}F^{(2)}(2)=-1, \qquad \frac{1}{3!}F^{(3)}(2)=3,$$

and

$$\frac{1}{1!}F^{(1)}(3)=\frac{1}{2!}F^{(2)}(3)=0, \qquad \frac{1}{3!}F^{(3)}(3)=1, \qquad \frac{1}{4!}F^{(4)}(3)=2, \qquad \frac{1}{5!}F^{(5)}(3)=1.$$

The submatrices of orders 2 and 3 which appear in the product M′PM are thus

$$\begin{bmatrix} 0 & -1 \\ -1 & 3 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix},$$

and their reciprocals are

$$\begin{bmatrix} -3 & -1 \\ -1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 3 & -2 & 1 \\ -2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Accordingly, equation (6) becomes

$$M^{-1}=\begin{bmatrix} -3 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -2 & 1 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 & 8 & 16 \\ 0 & 1 & 4 & 12 & 32 \\ 1 & 3 & 9 & 27 & 81 \\ 0 & 1 & 6 & 27 & 108 \\ 0 & 0 & 1 & 9 & 54 \end{bmatrix}\begin{bmatrix} 216 & -171 & 67 & -13 & 1 \\ -171 & 67 & -13 & 1 & 0 \\ 67 & -13 & 1 & 0 & 0 \\ -13 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} -135 & 216 & -126 & 32 & -3 \\ -54 & 81 & -45 & 11 & -1 \\ 136 & -216 & 126 & -32 & 3 \\ -84 & 136 & -81 & 21 & -2 \\ 36 & -60 & 37 & -10 & 1 \end{bmatrix}.$$

REFERENCES TO LITERATURE.

AITKEN, A. C., 1933. "On the Graduation of Data by the Orthogonal Polynomials of Least Squares," *Proc. Roy. Soc. Edin.*, vol. liii, pp. 54–78.

COURANT, R., and HILBERT, D., 1924. *Methoden der Mathematischen Physik* (Berlin), chap. ii.

FRAZER, R. A., DUNCAN, W. J., and COLLAR, A. R., 1938. *Elementary Matrices* (Cambridge), chap. iv.

FRAZER, R. A., JONES, W. P., and SKAN, S. W., 1937. "Approximations to Functions and to the Solutions of Differential Equations," *Reports and Memoranda of the Aeronautical Research Committee*, No. 1799.

(*Issued separately August 7, 1939.*)

XX.—Studies on Reproduction in the Albino Mouse. III. The Duration of Life of Spermatozoa in the Female Reproductive Tract.

By **Hugo Merton,** Institute of Animal Genetics, Edinburgh. *Communicated by* Professor F. A. E. Crew, F.R.S. (With Two Tables and Two Figures.)

(MS. received May 1, 1939. Read June 5, 1939.)

CONTENTS.

The method of artificial insemination in the mouse, which has been used successfully only once previously (Mark and Long, 1911), has made it possible to collect exact data on the duration of life and the fertilising capacity of spermatozoa in the female genital tract. Earlier results concerning the conditions under which spermatozoa from the male genital tract attain their maximum activity (Merton, II) and the exact knowledge of the time of parturition (Merton, I) were helpful in carrying out artificial insemination during the following œstrous period.

1*a*. Duration of the Upward Migration of the Spermatozoa in the Female Genital Tract; their Fertilising Ability and the Duration of Motility.

There exist several studies dealing with the upward migration of spermatozoa in the female genital tract after natural insemination, two of which may be mentioned here. According to Hartmann and Ball (1930), the spermatozoa of the rat became dispersed throughout the uterus 1–2 minutes following copulation. Lewis and Wright (1935), working on the mouse, found spermatozoa in the infundibular portion of the oviduct in one case 15 minutes, in another 40 minutes after ejaculation. "Either peristalsis or antiperistalsis, or even the motion of the

viscera would suffice to effect the churning up of the contents" (Hartman, 1932, p. 700).

If 0·1 c.c. of a coloured suspension is injected into the uterus of a mouse during the œstrus following parturition, dissection 10 minutes later shows it distributed throughout the entire uterus. If the latter organ is examined 1½ hours after injection, the dye can be seen accumulated at its tubal ends; a similar accumulation of the liquid can be observed after artificial insemination. Such aggregates are conspicuous owing to the admixture of a certain amount of blood from the sites of placentation. Whereas the first diffused distribution of the injected liquid may be due to mechanical causes, the proximal aggregates are undoubtedly the result of antiperistaltic waves of contraction of the uterine muscles. Rossmann (1937) observed in the rat uterine peristalsis 10 minutes after copulation during the œstrus in the course of a sexual cycle but rarely found antiperistalsis, though it must be remembered that at this stage the uterus is filled with liquid.

During the œstrus following parturition the uterus contains but little liquid, and its wall is always quite flaccid for the first 12 hours after parturition; in animals of my strains (Edinburgh) a marked increase in tone was observed only 14 hours post-partum. This change, however, is unrelated to the accumulation of liquid in the tubal end of the uterus, since it occurs also in animals inseminated artificially 11½ hours post-partum and examined 1½ hours later. Sperm were found regularly in the lowest loop of the oviduct only 1½ hours after insemination (compare Table I, No. 3)—that is, at a time when the anterior ends of the uteri into which the oviducts enter were somewhat distended by the accumulated contents.

The hour of birth was determined by means of the author's self-recording balance (Merton, 1938). The mouse stocks came partly from Edinburgh, partly from Yorkshire and the south of England. During January–April, œstrus following birth occurred comparatively late, no mating being observed before 20 hours post-partum. If at that time the ova had already reached the oviduct, they were still in the uppermost loop. (Since our object is to trace migration of the spermatozoa starting from the uterus, the six loops of the oviduct are numbered serially in the tables beginning from the uterine end; thus in the column headed "Sperm in Oviducts," No. 1 means the loop next to the uterus.) For insemination, use was made of a suspension of spermatozoa from the *vas deferens* and the *cauda epididymis* in Ringer's solution. In order to obviate in the female the possibility of changes in the interval between killing and microscopic examination, dissection was carried through so quickly that

in 5 minutes both oviducts had been separated from the uterus and one could be examined in a cover-slip preparation.

Two hours after artificial insemination (p.a.i.) some spermatozoa were always found in the vicinity of the discharged ova. After 2–3 hours one or two spermatozoa may easily be observed gliding in a swift winding motion over the surface of an ovum without losing contact with it. At this time the connection between the *zona pellucida* and follicular cells has already become severed through the action of the spermatozoa (*cf.* Yamane, 1935; Pincus and Enzmann, 1936). In the majority of the ova fixed 3½–4 hours after copulation and examined in sections the male pronucleus was found in the cytoplasm. Moreover, these ova showed the spindles for the second maturation division, and in one the second polar body had already been constricted off almost completely. If the spermatozoa were suspended in a mixture consisting of two parts Ringer solution and one part $\frac{m}{10}$ $NaHCO_3$, then during the first hour, as with pure Ringer, no spermatozoa entered the oviduct, but, on the other hand, after 1½ hours they were found between the third and fourth loop (Table I, Nos. 14, 15).

If transmission from the cervix to the upper oviduct depended solely on the forward movement of the spermatozoa by their own energy they ought to have reached their destination much earlier than was found to be the case. The uterine horns have an average length of 3·5 cm.—often one-half is even markedly shorter—and the Fallopian tubes an average length of 1·6 cm. But even in the tubes in which the spermatozoa swim upward by their own force, they stop repeatedly and for prolonged periods.

While it may be very laborious to decide exactly, through observation of coitus, the various stages which precede fertilisation, this is rendered much easier by using artificial insemination, which also makes possible the creation of experimentally suitable conditions, *e.g.* insemination before ovulation or much later. When carried out at a stage when there were no ova in the oviduct, the upward migration of the spermatozoa was not in any way influenced (Table I, Nos. 3, 7); the presence of ova is thus unnecessary for the upward migration of the spermatozoa. If any chemotactic substance should be excreted by the ova it is without importance for their directional movement. The time required to reach the fifth loop was always the same. When a.i. was carried out later than 31 hours post-partum mice of different strains reacted in slightly different ways. In one strain semen introduced 32 hours after parturition was expelled immediately from the uterus; in another, fertilisation occurred

even after 33 hours (Table I, Nos. 12, 13); and in a third, after 35 hours some of the ova were still fertilised.

TABLE I.—OCCURRENCE OF SPERM AND EGGS IN FEMALE GENITAL TRACT AFTER ARTIFICIAL INSEMINATION.

No.	Month of Year.	Inseminated Hours after Parturition.	Examined Hours after Insemination.	Eggs in Oviducts.	Sperm in Oviducts (Loop Number).	Sperm in Uterus.	Remarks.
1	III	23·30	0·50	+ +	–	+	Nos. 1–13, sperm in Ringer solution.
2	III	28·30	1·00	+ +	–	+	
3	III	11·30	1·30	– –	1	+	
4	III	28·20	2·00	+ +	5	+	
5	III	20·30	2·15	+ +	1, 5	+	
6	II	19·15	3·00	+ +	5	+	
7	II	13·30	3·00	– –	4, 5	+	
8	II	30·30	6·30	+ +	5	+	Eggs fertilised, 2 pronuclei.
9	III	18·15	13·30	+ +	2–5	+ dead	,, ,, ,,
10	IV	21·30	14·00	+ +	4, 5	+ dead	,, ,, ,,
11	III	19·15	15·00	+ +	–	–	Eggs fertilised.
12	II	32·10	2·20	+ –	–	–	Sperms expelled after insemination.
13	IV	33·00	3·00	+ +	3–5	+	
14	IV	23·45	1·00	+ –	–	+	Nos. 14 and 15, sperm in Ringer solution + $m/10$ $NaHCO_3$, 2 : 1.
15	IV	15·00	1·30	– –	2–4	+	

The figures in the column headed "Sperm in Oviducts" indicate in which loop of the duct (beginning from the uterine end) spermatozoa have been observed.

Especially in view of the possibility of artificial influences on the fertilisation process it appeared important to establish the duration of motility of the spermatozoa in the Fallopian tubes. This is rendered possible by the fact that their upward migration continues after the first of them have met the ova and fertilised them. It is particularly difficult to understand why at first only isolated spermatozoa migrate upwards through the Fallopian tubes, since, contrary to older data (Sobotta, 1895), both after copulation and after artificial insemination a considerable number of spermatozoa were found to enter the first lowest loop of the oviduct. This reservoir, which might be termed a *receptaculum seminis*, furnishes the spermatozoa migrating subsequently. From this it is possible to determine accurately the duration of their life in the female genital tract. On the basis of repeated observations, we consider that duration of life may be identical with duration of motility. When, for example, a Fallopian tube is examined 6½ hours p.a.i., 6–10 spermatozoa may be found in the expanded fifth loop containing the ova; in agreement with the findings of Sobotta, no spermatozoa were found farther

upward on the way to the infundibulum. Up to 13½ hours after insemination spermatozoa showing moderate activity could be found in the Fallopian tube (Table I, No. 9). At this stage two pronuclei were readily recognised in the fertilised ova. If the examination was carried out after 14 or more hours, spermatozoa were no longer visible in the total preparations of the Fallopian tube, since they had become immotile (Table I, Nos. 10, 11), but if it was dissected the dead spermatozoa could be recovered and were always in very good condition, in contrast to those from the uterus, which were for the most part without heads. It may be assumed that dead spermatozoa are carried back into the uterus and there degenerate.

In order to determine whether the intervals established were the result of artificial insemination with sperm suspended in Ringer's solution a number of controls were examined after copulation. The results (Table II) are in complete agreement with those for the artificial method: after 1½ hours spermatozoa had proceeded no farther up the Fallopian tube than in the former case, and 14¾ hours after coitus a number of dead spermatozoa were found in the vicinity of the ova in the ampullary region of the oviduct.

TABLE II.—OCCURRENCE OF SPERM AND EGGS IN FEMALE GENITAL TRACT FOLLOWING POST-PARTUM COITUS.

Serial No.	Month of Year.	Coitus after Parturition in Hours.	Examined Hours after Coitus.	Eggs in Oviducts.	Sperm in Oviducts.	Sperm in Uterus.
514	IV	28·00	0·15	+ +	-	+
516	IV	20·00	1·30	- -	1	+
513	IV	24·30	13·30	+ +	2, 4, 5	-
515	IV	27·30	14·45	+ +	4, 5	-

See explanation to Table I.

Since, firstly, motility of spermatozoa in the female genital tract persists for 13½ hours, and, secondly, on the average the ova reach the Fallopian tube 19–22 hours after parturition, the fertilising ability of the spermatozoa could be tested in a very simple way. In the present study it was found that in the period January–March insemination should not be carried out earlier than 18 hours after parturition in order to result in fertilisation. This means that the spermatozoa of the males used in this experiment retain their fertilising ability on the average for not more than about 6 hours after entering the female genital tract; on the basis of calculations alone the time is even shorter, but the fact must be taken into consideration that the ova in the fifth loop are so thickly

surrounded by follicular cells that it might take some time for the spermatozoa to penetrate to the surface of the ovum. Thus, the spermatozoa of the albino mouse are already losing their fertilising ability towards the end of the first half of their period of motility in the female genital tract. In order to be sure that fertilisation had actually occurred the animals were examined 48 hours p.a.i. In positive cases the ova were in the four-cell stage.

1*b*. DISCUSSION.

It can be shown that during an ordinary œstrous cycle the conditions in the uterus of the mouse differ from those during œstrus following parturition. In the first case the uterus is completely filled with liquid for several hours, and within a short time the spermatozoa reach the opening of the Fallopian tube, partly by active, partly by passive, movement; according to Lewis and Wright (1935) they may reach the ova within the first hour. In the second (where the sperm reach the eggs at the end of 2 hours) the uterus contains but little fluid. Though here also spermatozoa are found in all parts of the uterus after a few minutes, they reach the Fallopian tube only when, through an antiperistaltic wave of contraction after 1½ hours, a considerable number of them have collected in the anterior region. It should be kept in mind that this time is quite sufficient for a considerable number of spermatozoa to travel the length of the uterus by their own force. The delayed entrance of the spermatozoa into the Fallopian tube may perhaps be explained by the suggestion that the papilla of the tube protrudes somewhat into the uterus and opens only when it is surrounded by liquid. It is improbable that the posteriorly directed cleansing process after parturition delays the migration of the spermatozoa, because in this case the degree of delay would have differed with the time of insemination. It may be mentioned here that in the rabbit the passage of the spermatozoa through the uterus, which is poor in liquid (Hartman, 1932), takes 2 hours, and is explained by assuming that the spermatozoa cover this distance by their own energy (Parker, 1931). It would be interesting to verify observations reported here for the mouse on other mammals.

The behaviour of the uterus and Fallopian tube towards the spermatozoa differs; the uterine fluid was shown to prolong life in experiments *in vitro* (Loew, 1902; Kugota, 1929), and the uterine mucosa to exercise a strong chemotactic influence (Loew). These properties are, however, of secondary importance for fertilisation in the mouse and rat, since the spermatozoa concerned in fertilisation have already reached the Fallopian tube 1½ hours after artificial insemination. According to Exner (1904),

no difference exists between the chemotaxis of the spermatozoa in the uterus and the oviduct, but conditions in the oviduct are markedly more favourable for the spermatozoa. Popa and Marza (1929), who worked on the guinea-pig and the dog, claim that the degree of sperm degeneration increases with their spatial relationship to the ovary, the maximum being reached at the infundibulum; these conclusions, however, require confirmation. According to the author's observations on the mouse, the spermatozoa in the Fallopian tube cease moving only after the whole store of energy has been used up, which, of course, occurs earlier at body than at room temperature. No indications of defensive reactions of the oviduct in respect of the spermatozoa were found in the albino mouse, though Charlton (1917) states that in the lowest portion of the oviduct phagocytes co-operate in the removal of unfertilised ova.

According to the experiments reported here the actual fertilisation in the mouse occurs in the course of the third hour after insemination in the œstrus following parturition; according to Lewis and Wright (1935) it occurs a little earlier, for they observed the expulsion of the second polar body 2½–5 hours after copulation. According to Mark and Long (1911) the time between artificial insemination during the œstrus cycle and fertilisation varies from 4–7 hours, while Sobotta (1895) states that the spermatozoa enter the ova only 6–10 hours after copulation. It is possible that it may differ according to strain, but with an interval of more than 5 hours an error of observation cannot be excluded. Pincus and Enzmann (1932) stated that in the rabbit the critical period for sperm penetration in the egg occurs at 2–3 hours after ovulation, which, as is well known, takes place 10 hours after copulation.

It could be shown that in the mouse the spermatozoa retain their fertilising ability only during the first half of the period during which they exhibit activity in the female genital tract. Similar observations have been reported for other mammals. In the rabbit the spermatozoa retain their fertilising ability for not more than 30 hours, but motility persists considerably longer (Hammond and Asdell, 1927). In the genital tract of the sow the spermatozoa die 20–24 hours after copulation, and their period of fertilising ability is even shorter (Lewis, 1911; Haring, 1937).

2*a*. The Duration of Life of the Spermatozoa in the Uterus and Phagocytosis.

Only a very small percentage of the spermatozoa which are present in the uterus of the mouse after copulation reach the Fallopian tubes. Observations to determine whether the fate of the spermatozoa remaining in the uterus during the œstrus after parturition is the same as during

the ordinary œstrus cycle show that this is not the case. Copulation after parturition may or may not be accompanied by the formation of a vaginal plug, but even if the plug is present it will close the uterus for only a short time, since the extrusion through the *cervix uteri* of blood and shreds of tissue expelled from the previous placentations renders the plug ineffective. As has been shown above, this does not influence fertilisation. Incidentally the softening of the plug in females which have been fertilised during the ordinary œstrous cycle may begin as early as 10–12 hours after fertilisation (Sobotta, 1911), and the plug may be expelled after widely varying intervals (Lewis and Wright, 1933).

The migration of the leucocytes into the uterus after mating or artificial insemination following parturition is less marked than after copulation during other œstrous cycles, and phagocytosis also occurs to a much lesser degree. Approximately 40 per cent. of the spermatozoa in the uterus were still active 6½ hours after artificial insemination; 9–10 hours after copulation all spermatozoa had ceased movement, and approximately 4–5 hours later no more spermatozoa or leucocytes were present in the uterus; similar data were obtained following artificial insemination.

Spermatozoa, removed from the uterus 3–4 hours after copulation and transferred to a solution of glucose in Ringer, retained their motility up to 24 hours, a few even up to 30 hours. These spermatozoa had a sticky surface, as shown by the fact that they aggregated in small masses and could not free themselves. Due to this agglutination, motility was slowed down considerably and energy consumption reduced. At any rate it is clear that during the first hours in the uterus they do not suffer to any considerable extent.

For reasons given above, females which have been served during the œstrous cycle must be used for the study of phagocytosis. When the mice are examined at different intervals after copulation it is seen that phagocytosis becomes more marked 8 hours, and ceases about 16–17 hours, after copulation. The bulk of the spermatozoa are found in thick clumps in the uterine lumen; in agreement with the findings of Königstein (1908) and Sobotta (1920) these were not attacked by leucocytes, the action of the latter being restricted to the destruction of active spermatozoa swimming forwards between the central mass of sperm and the wall of the uterus, or of isolated ones which have just stopped moving. Leucocytes, which had just penetrated the wall of the uterus, were seen in small inactive colonies, but subsequently they became dispersed. The quickly-moving spermatozoa have to pass the leucocytes; as soon as they touch them, the point of contact adheres to the leucocyte and their advance is prevented. If the spermatozoon adheres to the leucocytes, not with

the head, but with the middle-piece or tail, the leucocytes put out two tentacle-like pseudopodia which move in opposite directions along it; on meeting, they contract and drag part of the spermatozoon into the cell body (fig. 1); eventually the entire spermatozoon is engulfed, and can be seen there as a ball-like inclusion (figs. 1, 2). In view of the elasticity of the spermatozoa this process requires a considerable amount of energy. The spermatozoa of the mouse are approximately twelve times as long ($123\ \mu$) as the diameter of a leucocyte. The process described here can be seen with particular clearness in degenerating leucocytes

FIG. 1.—Leucocyte engulfing a spermatozoon. Two pseudopodia have fused together; the spermatozoon is already partly inside the cell body. In two of the other three leucocytes phagocytosis has already taken place.

FIG. 2.—Two leucocytes attacking the same spermatozoon, one at each end. The elongated leucocyte below has devoured an entire spermatozoon.

and resembles the manner of feeding of *Amœba verrucosa* when it swallows long threads of *Oscillaria*; the best permanent preparations are obtained by fixing a smear with methyl alcohol, drying, and staining.

The partially ingested spermatozoon still continues to exhibit vibratile movement for some time, and in those in which phagocytosis begins at the head the greater part of the middle-piece is swallowed before movement stops. Until now it had been assumed that only sperm heads undergo phagocytosis (Sobotta, 1920) as only these were found inside the leucocytes, but this observation now finds its explanation in the fact that after the whole spermatozoon has been digested only the head with its undigestible capsule remains. As many as three sperm heads may be found in one leucocyte. One leucocyte may devour two or three spermatozoa simultaneously, but sometimes, on the other hand, two leucocytes may attack one spermatozoon, one at each end (fig. 2). 23 hours after copulation (the average time required for uterine cleansing)

hardly any spermatozoa can be found in the uterus apart from some sperm heads within or outside the leucocytes. The spermatozoa which were not phagocytosed have dissolved completely.

2*b*. Discussion.

Only a detailed study of the life habits of those wild types from which the domesticated breeds have been derived will reveal the significance of the plug which closes the vagina after copulation in various rodents (*e.g.* mouse, rat, guinea-pig). In the mouse the plug is of no importance for successful fertilisation, but Cooley and Slonacker (1925) found that only rats with a well-formed plug became pregnant.

In contrast to the Fallopian tube, the uterus mobilises various forces of defence very soon after the entrance of the spermatozoa. Already during the 4th hour after copulation the number of active spermatozoa begins to decrease, and after the 6th hour many of them have ceased to move. In females mated during the ordinary sexual cycle some spermatozoa remain motile for a longer time (up to 16 hours) than in post-parturient females (9 hours). Sobotta (1911) observed a shorter duration of life of the spermatozoa in the uterus, and the results differed considerably according to the strain. On the average the spermatozoa of the mouse survive in the uterus for a somewhat shorter period than do those of the rat. In the latter Yochem (1929) observed movement of spermatozoa injected into the uterus 12 hours after injection, and following copulation there were still some motile spermatozoa 17½ hours later.

In females served during ordinary cycles phagocytosis is much more pronounced than in post-parturient females. Sobotta (1920) underestimates the importance of phagocytosis when he states that it occurs only when a spermatozoon has spontaneously penetrated into a leucocyte. Königstein (1908), on the basis of his observations on rats, ascribes a greater significance to this phenomenon; he assumes, however, that phagocytosis starts only after motility has ceased. As a result of the present studies it has been shown that a considerable number of active spermatozoa become ingested in the space between the wall of the uterus and the central mass of sperm. In contrast to previous statements, it has been established that not only the heads but entire spermatozoa became engulfed.

Acknowledgment.

Practically all the studies reported here were carried on in the Institute of Animal Genetics at Edinburgh, and were made possible by the kind hospitality of Professor F. A. E. Crew and a grant from the Society for the Protection of Science and Learning.

3. SUMMARY.

1. After copulation or artificial insemination during the œstrus following parturition, spermatozoa enter the oviduct only after liquid has accumulated at the proximal ends of the uterus.

2. Spermatozoa enter the Fallopian tubes after 1½ hours; they meet the ova at the end of the 2nd hour, and fertilisation occurs in the course of the 3rd hour.

3. After artificial insemination the spermatozoa may migrate upwards in the tubes even though the eggs enter them only 8–10 hours later.

4. In the Fallopian tubes the spermatozoa retain their fertilising ability for about 6 hours; motility ceases only after 13½ hours.

5. During œstrus following parturition, fertilisation may take place without the formation of the vaginal plug. The degenerated spermatozoa are expelled from the uterus, on the average, after 14 hours.

6. The phagocytes in the uterus are capable of engulfing whole spermatozoa, which later are found included in the cell body.

7. Only living or newly dead spermatozoa are phagocytosed; the head capsule is not digested, and eventually becomes extruded.

4. REFERENCES TO LITERATURE.

CHARLTON, H., 1917. "The Fate of the Unfertilised Egg in the White Mouse," *Biol. Bull.*, vol. xxxiii, pp. 21–29.

COOLEY, C. L., and SLONAKER, F. R., 1925. "The Effects of Early and Late Breeding on the Mother and the Sex Ratio in the Albino Rat," *Amer. Journ. Physiol.*, vol. lii, p. 595.

EXNER, S., 1904. "Physiologie der männlichen Geschlechtsfunktionen," *v. Frisch und Zuckerhandl—Handbuch d. Urologie*, vol. i, p. 244.

HAMMOND, J., and ASDELL, S. A., 1927. "The Vitality of the Spermatozoa in the Male and Female Reproductive Tracts," *Brit. Journ. Exp. Biol.*, vol. iv, pp. 155–185.

HARING, F., 1937. "Der Einfluss des Begattungszeitpunkts während der Rauschzeit beim Schwein auf den Befruchtungserfolg und das Geschlechtsverhältnis der Ferkel," *Züchtungskunde*, vol. xii, pp. 1–18.

HARTMAN, C. G., and BALL, J., 1930. "On the Almost Instantaneous Transport of Spermatozoa through the Cervix and the Uterus in the Rat," *Proc. Soc. Exp. Biol. Med.*, vol. xxviii, pp. 312–314.

HARTMAN, C. G., 1932. "Ovulation and the Transport and Viability of Ova and Sperm in the Female Genital Tract," *E. Allen, Sex and Internal Secretions*, Baltimore, p. 647.

KNAUS, H., 1932. "Zur Physiologie der Spermatozoen," *Arch. Gynäk.*, vol. cli, pp. 302–323.

KÖNIGSTEIN, H., 1908. "Uber das Schicksal der Spermatozoen, welche nicht zur Befruchtung gelangen," *Wien. Klin. Wochenschr.*, vol. xxi.

KUGOTA, T., 1929. "Der Einfluss des Uterussaftes in verschiedenen Perioden des oestrischen Cyklus auf die Lebensdauer der Spermatozoen, *Zeits. Zellf. mikr. Anat.*, vol. ix, pp. 457–464.

LEWIS, W. H., and WRIGHT, E. ST., 1935. "On the Early Development of the Mouse Egg," *Publ. Carnegie Inst.*, No. 459, pp. 113–144.

LEWIS, L. L., 1911. "The Viability of Reproductive Cells," *Bull. Oklohama Agric. Michigan Coll.*, vol. xcvi.

LOEW, O., 1902. "Die Chemotaxis der Spermatozoen im weiblichen Geschlechtstrakt," *Sitzb. Akad. Wiss. Wien. Math. Naturw.*, Kl. iii, vol. cxi.

MARK, E. L., and LONG, F. A., 1911. "The Maturation of the Egg of the Mouse," *Publ. Carnegie Inst.*, No. 142, pp. 1–72.

MERTON, H., 1938. "Studies on Reproduction in the Albino Mouse. I. The Period of Gestation and the Time of Parturition," *Proc. Roy. Soc. Edin.*, vol. lviii, pp. 80–97.

——, 1939. "Studies on Reproduction in the Albino Mouse. II. Contributions on the Maturation of the Sperm Cell," *Proc. Roy. Soc. Edin.*, vol. lix, pp. 145–152.

PARKER, G. H., 1931. "The Passage of Sperms and of Eggs through the Oviducts in Terrestrial Vertebrates," *Phil. Trans. Roy. Soc. London*, B, vol. ccxix, pp. 381–419.

PINCUS, G., and ENZMANN, E. V., 1932. "Fertilisation in the Rabbit," *Journ. Exp. Biol.*, vol. ix, pp. 403–408.

—— ——, 1936. "The Comparative Behaviour of Mammalian Egg *in vivo* and *in vitro*. II. The Activation of Tubal Eggs in the Rabbit," *Journ. Exp. Zool.*, vol. lxxiii, pp. 195–208.

POPA, GR. F., et MARZA, V., 1929. "La phagocytose des spermatozoides vivants par les éléments cellulaires du tractus génital femelle de la même espèce," *C.R. Soc. Biol.*, vol. ci, pp. 1185–1186.

ROSSMANN, J., 1937. "Uterine Contractions and the Transport of Sperm in the Rat," *Anat. Record*, vol. lxxix, pp. 133–149.

SOBOTTA, F., 1895. "Die Befruchtung und Furchung des Eies der Maus," *Arch. mikr. Anat.*, vol. xlv.

——, 1911. "Uber des Verhalten der Spermatozoen im Uterus der Säugetiere, nach den Befunden bei Nagetieren," *Zeits. Morph. Anthropol.*, vol. xiii, pp. 201–208.

——, 1920. "Was wird aus den in den Uterus ejakulierten und nicht zur Befruchtung verwendeten Spermatozoen," *Arch. mikr. Anat.*, vol. xciv.

YAMANE, J., 1935. "Kausalanalytische Studien über die Befruchtung des Kanincheneies. I. Die Dispersion der Follikelzellen und die Ablösung der Corona radiata des Eies durch Spermatozoen," *Cytologia*, vol. vi, pp. 233–255.

YOCHEM, D. E., 1929. "Spermatozoon Life in the Female Reproductive Tract of the Guinea-Pig and the Rat," *Biol. Bull.*, vol. lvi, pp. 274–297.

(*Issued separately September* 14 1939.)

XXI.—Reciprocity and the Number 137. Part I. By Professor Max Born, F.R.S., University of Edinburgh.

(MS. received July 12, 1939.)

INTRODUCTION.

PROFESSOR A. LANDÉ has kindly sent me a manuscript of a paper* in which he develops some most important ideas on the meaning and the value of the number 137. His work is founded on the principle of reciprocity which I recently suggested (Born, 1938). Landé adds to this principle a new assumption about the wave functions, representing an electron, and he finds a linear integral equation for these containing a parameter the value of which may be expected to be 137. It is Landé's great merit to have shown that the reciprocity principle leads to a condition for the Planck constant $\hbar$ (the value of which in natural units, $c=1$, $e=1$, is 137), and to have discovered the root of the peculiar order of magnitude of the number. But some difficulties which I encountered led me to reconsider the problem.

I shall show that a direct application of the principle of reciprocity leads to the existence of wave functions representing a resting particle which satisfy a linear homogeneous integral equation of the normal (Fredholm) type containing the parameter λ linearly. Although a complete theory has to reformulate the electrodynamic laws in connection with reciprocity, one can safely assume that the expression for the electrostatic energy of a resting charge distribution will hold unchanged. Doing this, one gets a definite expression for $\hbar c/e^2$ in terms of the eigenvalue λ and an integral over the corresponding eigenfunction. The numerical evaluation will take some time and will be given later. Until then the theory has to be considered only as a mathematically attractive suggestion.

1. *Formulation of the Assumptions.*—We connect the units of space and time in such a way that $c=1$. The principle of reciprocity postulates that to each equation in the x-space there corresponds a similar equation in the p-space. We consider a particle moving freely with energy E and momentum $\mathbf{p}$; then

$$E^2 - p^2 = \epsilon^2, \qquad \text{(A)}$$

where ϵ is the rest energy. The reciprocal equation is

$$t^2 - r^2 = \tau^2, \qquad \text{(B)}$$

* To be published in an American periodical.

where τ is a constant connected with the particle in the same way as ϵ. Equation (B) represents a law of retardation which differs from that for light waves, namely $t^2 - r^2 = 0$ (Dirac, 1938). If one intersects the hyperbolic spaces (B) by the planes $t=$const. one gets spheres expanding with the velocity of light for large t, but with higher velocity if t is approaching τ. Values of $|t|$ smaller than τ are forbidden by (B); in complete analogy to the well-known fact that values of $|E| < \epsilon$ are excluded.

This behaviour of time of retardation seems strange; but we are building a new wave kinematics and have to accept new conceptions if they lead to a coherent theory.

Now consider a wave packet in x-space ($h = 2\pi\hbar$):

$$\psi(\mathbf{r}, t) = \frac{1}{\sqrt{h^3}}\int \Phi(\mathbf{q}) e^{\frac{i}{\hbar}(\mathbf{r}\cdot\mathbf{q} - \sqrt{\epsilon^2+q^2}\,t)} d\bar{q}, \qquad d\bar{q} = dq_x dq_y dq_z, \qquad (1)$$

where we have written q (instead of p) for the integration variable, in order to have p available for the reciprocal formula; this is evidently (ρ is the vector ρ_x, ρ_y, ρ_z in x-space):

$$\phi(\mathbf{p}, E) = \frac{1}{\sqrt{h^3}}\int \Psi(\rho) e^{-\frac{i}{\hbar}(\rho\cdot\mathbf{p} - \sqrt{\tau^2+\rho^2}\,E)} d\bar{\rho}. \qquad (2)$$

The essential point is this: The moving wave packet (1) in x-space is described in the p-space by a function $\Phi(\mathbf{q}) = \Phi(q_x, q_y, q_z)$ which is independent of E; we can call it a "static" state in p-space. In the same way a general wave packet in p-space belongs to a "static" function $\Psi(\rho) = \Psi(\rho_x, \rho_y, \rho_z)$ in x-space.

The two pairs of functions ψ, Φ and ϕ, Ψ are independent so far; but we connect them now by a new postulate. (Landé has the merit of having seen the necessity of such a postulate, but his formulation is essentially different.)

The postulate of particles at rest is

$$\psi(\mathbf{r}, \sqrt{\tau^2+r^2}) = \Psi(\mathbf{r}), \qquad \phi(\mathbf{p}, \sqrt{\epsilon^2+p^2}) = \Phi(\mathbf{p}). \qquad (3)$$

The interpretation is this: In order that a wave packet may represent a particle at rest, the ψ-function in x-space taken for times properly retarded —namely, on the hyperboloid $t = \sqrt{\tau^2+r^2}$ (not on the light-cone)—must be identical with the Ψ-function which is the Fourier coefficient of the same wave packet in the p-space; and *vice versa*.

2. *The Integral Equation.*—From (3) follows by substitution from (1) and (2):

$$\Psi(\mathbf{r}) = \psi(\mathbf{r}, \sqrt{\tau^2 + r^2}) = \frac{1}{\sqrt{h^3}} \int \Phi(\mathbf{q}) e^{\frac{i}{\hbar}(\mathbf{r} \cdot \mathbf{q} - \sqrt{\epsilon^2 + q^2} \cdot \sqrt{\tau^2 + r^2})} d\bar{q}$$

$$= \frac{1}{\sqrt{h^3}} \int \phi(\mathbf{q}, \sqrt{\epsilon^2 + q^2}) e^{\frac{i}{\hbar}(\mathbf{r} \cdot \mathbf{q} - \sqrt{\epsilon^2 + q^2} \cdot \sqrt{\tau^2 + r^2})} d\bar{q}$$

$$= \frac{1}{h^3} \int d\bar{q} \int d\bar{\rho}\, \Psi(\rho) e^{\frac{i}{\hbar}(\mathbf{r} \cdot \mathbf{q} - \rho \cdot \mathbf{q} - \sqrt{\epsilon^2 + q^2} \cdot \sqrt{\tau^2 + r^2} + \sqrt{\tau^2 + \rho^2} \cdot \sqrt{\epsilon^2 + q^2})}.$$

This can be written in the short form:

$$\Psi(\mathbf{r}) = \int d\bar{\rho} \,.\, \mathbf{K}_{\tau\epsilon}(\mathbf{r}, \rho) \,.\, \Psi(\rho), \qquad (4)$$

where

$$\mathbf{K}_{\tau\epsilon}(\mathbf{r}, \rho) = \frac{1}{h^3} \int d\bar{q}\, e^{\frac{i}{\hbar}\{(\mathbf{r} - \rho) \cdot \mathbf{q} - (\sqrt{\tau^2 + r^2} - \sqrt{\tau^2 + \rho^2})\sqrt{\epsilon^2 + q^2}\}}. \qquad (5)$$

In the same way we find for $\Phi(\mathbf{p})$:

$$\Phi(\mathbf{p}) = \int d\bar{q} \,.\, \mathbf{K}^*_{\epsilon\tau}(\mathbf{p}, \mathbf{q}) \,.\, \Phi(\mathbf{q}), \qquad (6)$$

where the star means the conjugate complex quantity.

(4) is a linear integral equation and (6) its conjugate. The kernel $\mathbf{K}_{\tau\epsilon}(\mathbf{r}, \rho)$ is hermitean:

$$\mathbf{K}_{\tau\epsilon}(\mathbf{r}, \rho) = \mathbf{K}^*_{\tau\epsilon}(\rho, \mathbf{r}). \qquad (7)$$

We introduce dimensionless variables by choosing in (4) and (5):

$$\mathbf{r} = \tau\xi, \qquad \rho = \tau\eta, \qquad \mathbf{q} = \epsilon\sigma,$$

in (6)

$$\mathbf{p} = \epsilon\xi, \qquad \mathbf{q} = \epsilon\eta,$$

and

$$\frac{\tau\epsilon}{\hbar} = \frac{2\pi\tau\epsilon}{h} = \mu. \qquad (8)$$

Then we get from (5):

$$\mathbf{K}_{\tau\epsilon}(\mathbf{r}, \rho) = \left(\frac{\epsilon}{h}\right)^3 \mathrm{K}(\xi, \eta), \qquad \mathbf{K}^*_{\epsilon\tau}(\mathbf{p}, \mathbf{q}) = \left(\frac{\tau}{h}\right)^3 \mathrm{K}^*(\xi, \eta), \qquad (9)$$

where

$$\mathrm{K}(\xi, \eta) = \int d\bar{\sigma}\, e^{i\mu\{(\xi - \eta) \cdot \sigma - (\sqrt{1 + \xi^2} - \sqrt{1 + \eta^2})\sqrt{1 + \sigma^2}\}}, \qquad (10)$$

and the integral equations (4) and (6) are equivalent to

$$f(\xi) = \lambda \int d\bar{\eta} \,.\, \mathrm{K}(\xi, \eta) \,.\, f(\eta) \qquad (11)$$

and its complex conjugate, if one substitutes for (4):

$$\left.\begin{aligned} \Psi(\mathbf{r}) &= \mathrm{A}f(\xi), \quad \text{if} \quad \mathbf{r} = \tau\xi, \\ \Phi(\mathbf{p}) &= \mathrm{B}f^*(\xi), \quad \text{if} \quad \mathbf{p} = \epsilon\xi. \end{aligned}\right\} \qquad (12)$$

for (6):

The parameter λ has the meaning $(\epsilon\tau/\hbar)^2 = (\mu/2\pi)^2$; each eigenvalue is a definite function of μ, $\lambda(\mu)$; therefore one has

$$\lambda(\mu) = (\mu/2\pi)^2. \qquad (13)$$

(11) is an ordinary integral equation of the Fredholm type, for given μ linear in the parameter. Its solution gives the eigenvalues λ as functions of μ, and then μ has to be determined from (13).

The kernel is a kind of "finite δ-function," as is to be expected in a theory where particles have finite dimensions. By omitting the square roots in the exponent in (10) one gets an ordinary 3-dimensional δ-function (the product of 3 simple δ-functions for the components of the vector $\xi - \eta$). By introducing polar co-ordinates in the σ-space one can easily perform the angular integrations and obtain:

$$\mathrm{K}(\xi, \eta) = \frac{4\pi}{\mu\sqrt{\xi^2 + \eta^2 - 2(\xi \cdot \eta)}} \int_0^\infty \sin\left(\mu\sigma\sqrt{\xi^2 + \eta^2 - 2(\xi \cdot \eta)}\right) e^{-i\mu(\sqrt{1+\xi^2} - \sqrt{1+\eta^2}) \cdot \sqrt{1+\sigma^2}} \sigma d\sigma. \qquad (14)$$

If we have to do with one isolated particle we shall have

$$\int |\Psi(\mathrm{r})|^2 d\bar{r} = 1, \qquad \int |\Phi(\mathrm{p})|^2 d\bar{p} = 1, \qquad (15)$$

hence for $\mathrm{A} = 1/\tau^{3/2}$, $\mathrm{B} = 1/\epsilon^{3/2}$:

$$\int |f(\xi)|^2 d\bar{\xi} = 1. \qquad (16)$$

Although the definite formulæ can only be established after having the complete electrodynamics it is very likely that the following procedure is correct.

We calculate (with Landé) the electrostatic self-energy and identify it with ϵ:

$$\tfrac{1}{2}e^2 \iint \frac{|\Psi(\mathrm{r}_1)|^2 \cdot |\Psi(\mathrm{r}_2)|^2}{|\mathrm{r}_1 - \mathrm{r}_2|} d\bar{r}_1 d\bar{r}_2 = \epsilon. \qquad (17)$$

Introducing the numerical integral

$$\tfrac{1}{2} \iint \frac{|f(\xi_1)|^2 \cdot |f(\xi_2)|^2}{|\xi_1 - \xi_2|} d\bar{\xi}_1 d\bar{\xi}_2 = \gamma, \qquad (18)$$

the equation (17) becomes with (8)

$$e^2\gamma = \epsilon\tau = \hbar\mu = h\mu/2\pi. \qquad (19)$$

Our theory will, therefore, be confirmed if the solution of the integral equation leads to the existence of one definite value μ for which

$$\frac{\hbar}{e^2} = \frac{\gamma}{\mu} = 137. \qquad (20)$$

The mathematical theory of the integral equation will be given later. But I presume that the correct value for $\hbar/e^2$ will not be obtained, since we have treated the wave function as a scalar: the real wave function will be a spinor, and there will be a magnetic term in the energy equation.

Summary.

Following an idea of Landé, a generalised wave mechanics is developed which is suited to describe the existence of particles with finite dimensions. The quantum constant divided by the product of an absolute energy and an absolute time appears as the parameter in a linear homogeneous integral equation of the Fredholm type, and the numerical determination of the number "137" is reduced to the calculation of an eigenvalue.

References to Literature.

Born, M., 1938. *Proc. Roy. Soc.*, A, vol. clxv, p. 291; vol. clxvi, p. 552.

Dirac, P. A. M., 1938. *Proc. Roy. Soc.*, A, vol. clxvii, p. 148.

(*Issued separately September* 14 1939.)

XXII.—Integration of a Certain System of Linear Partial Differential Equations of Hypergeometric Type.

By **A. Erdélyi**, Mathematical Institute, University of Edinburgh. *Communicated by* Professor E. T. WHITTAKER, F.R.S.

(MS. received June 14, 1939. Read July 3, 1939.)

INTRODUCTION.

1. THE integration of systems of linear partial differential equations of hypergeometric type has been the subject of a great number of recent investigations (*e.g.* Appell, 1925; Appell-Kampé de Fériet, 1926). Especially Professor Horn (see references) and his pupils, for instance Borngässer (1933), have done much valuable work in this field.

The first method that suggests itself for integrating systems of this type is to try power-series expansions of the solutions in the neighbourhood of the singular points of the system. This method, however, does *not* yield in general a complete set of solutions. Those singular points of the system in which more than n singular manifolds intersect, n being the number of independent variables, give rise to considerable difficulties.

Every system of linear partial differential equations of hypergeometric type is in general reducible to a completely integrable system of total linear differential equations of first order (*e.g.* Appell-Kampé de Fériet, 1926, pp. 44–49, 54; Horn, 1931, pp. 394 *et seq.*), the number of *dependent* variables in this system giving the number of linearly independent solutions of the original equations. From this completely integrable system of *total* differential equations some more solutions can be obtained.

Another very powerful method of dealing with functions of several variables is to replace the set of variables by a set of arbitrary, *e.g.* linear, functions of only *one* auxiliary variable. Doing so, our system of partial differential equations reduces to a system of *ordinary* differential equations, the integration of which can be effected by well-known methods.. In fact, proceeding in this manner, further results could be attained (Horn, 1935, 1936, 1938 *a*, *b*).

2. It seems that none of the methods mentioned hitherto is capable of furnishing us in every case with a complete set of solutions and—the importance of this second point of view hardly needs to be emphasised—with the transformation scheme of solutions. There is, however, another

method which I believe to be of great value for integrating systems of partial linear differential equations.

Using a suitable functional transformation, every hypergeometric function can be "factorised." That is to say, every hypergeometric function of *several* variables $x_1, x_2, \ldots, x_n$ can be represented by an integral in the integrand of which only a product of functions of one variable each occur.

Take, for instance, Lauricella's series (Lauricella, 1893; see also Appell-Kampé de Fériet, 1926, chap. vii):

$$F_A(\alpha;\ \beta_1, \ldots, \beta_n;\ \gamma_1, \ldots, \gamma_n;\ x_1, \ldots, x_n)$$
$$= \sum_{m_1=0}^{\infty} \ldots \sum_{m_n=0}^{\infty} \frac{(\alpha)_{m_1+\ldots+m_n}(\beta_1)_{m_1} \ldots (\beta_n)_{m_n}}{(\gamma_1)_{m_1} \ldots (\gamma_n)_{m_n} m_1! \ldots m_n!} x_1^{m_1} \ldots x_n^{m_n}.* \qquad (1)$$

It can be factorised by Laplace's transform, for (Erdélyi, 1936)

$$F_A(\alpha;\ \beta_1, \ldots, \beta_n;\ \gamma_1, \ldots, \gamma_n;\ x_1, \ldots, x_n)$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-t} t^{\alpha-1} {}_1F_1(\beta_1;\ \gamma_1;\ x_1 t) \ldots {}_1F_1(\beta_n;\ \gamma_n;\ x_n t) dt \qquad (2)$$

if $\mathbf{R}(\alpha) > 0$ and all the x_i $(i = 1, \ldots, n)$ are sufficiently small; for other values of α a similar expression holds, the integral being replaced by the corresponding loop-integral. Replacing Kummer's series

$${}_1F_1(\beta;\ \gamma;\ w) = \sum_{m=0}^{\infty} \frac{(\beta)_m}{(\gamma)_m} \frac{w^m}{m!} \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

in (2) by the *general solution* of its differential equation

$$w\frac{d^2z}{dw^2} + (\gamma - w)\frac{dz}{dw} - \beta z = 0, \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

the integral obtained thus from (2) will still satisfy the same system of differential equations as (2) itself, and so we have the general solution of the system of linear partial differential equations:

$$x_j\frac{\partial^2 z}{\partial x_j^2} - x_j\sum_{k=1}^{n} x_k \frac{\partial^2 z}{\partial x_j \partial x_k} + [\gamma_j - (\alpha+1)x_j]\frac{\partial z}{\partial x_j} - \beta_j \sum_{k=1}^{n} x_k \frac{\partial z}{\partial x_k} - \alpha\beta_j z = 0\ . \qquad (5)$$
$$(j = 1, \ldots, n),$$

of which F_A is a particular solution.† Moreover, the application of the

* I accept the usual notation

$$(\alpha)_0 = 1,\quad (\alpha)_m = \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)} \begin{cases} = \alpha(\alpha+1) \ldots (\alpha+m-1) & \text{if } m = 1, 2, \ldots, \\ = \dfrac{1}{(\alpha-1)(\alpha-2) \ldots (\alpha+m)} & \text{if } m = -1, -2, \ldots \end{cases}$$

† I mentioned some consequences of this simple theorem in Erdélyi (1939 *a*). A more detailed discussion of the integration of (5) on these lines I hope to give in a future paper.

transformation theory of the solutions of the confluent hypergeometric differential equation (4) in the integrand of (2) furnishes us at once with the transformation theory of the solutions of (5).

The method, of which the preceding paragraph gives only a rough sketch, is by no means restricted to systems of *hypergeometric* type. It applies to a certain class of systems of linear partial differential equations. Its clue is the application of a suitable functional transformation which "separates" the independent variables. In the general case Professor Whittaker's method of solving differential equations by means of families of definite integrals (Whittaker, 1931) may be useful in choosing the appropriate functional transformation. The discussion of this may be left to a future paper.

3. In the case of the system of differential equations belonging to F_A, dealt with in the preceding section, the various integrals satisfying (5) differ from each other by their *integrands*. This is by no means the general rule. The "factorising" functional transformation often yields integral representations of the solutions of a certain system of linear partial differential equations differing from each other by having different *paths* of integration.

Take, for instance, the confluent hypergeometric series of several variables (Erdélyi, 1937 *b*, equation 7, 2):

$$ {}_n\Phi(\beta_1, \ldots, \beta_n;\ \gamma;\ x_1, \ldots, x_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}. \tag{6} $$

This function can also be "factorised" by Laplace transform, yielding the integral representation (*ibid.*, equation 8, 6):

$$ {}_n\Phi(\beta_1, \ldots, \beta_n;\ \gamma;\ x_1, \ldots, x_n) = \frac{\Gamma(\gamma)}{2\pi i}\int_L e^s s^{-\gamma}\left(1-\frac{x_1}{s}\right)^{-\beta_1} \cdots \left(1-\frac{x_n}{s}\right)^{-\beta_n} ds, \tag{7} $$

in which L denotes a path coming from $-\infty$, encircling in the positive direction all finite singularities of the integrand and returning to $-\infty$. We get a complete set of solutions of the system (*ibid.*, equation 7, 3):

$$ \sum_{k=1}^{n} x_k \frac{\partial^2 z}{\partial x_j \partial x_k} + (\gamma - x_j)\frac{\partial z}{\partial x_j} - \beta_j z = 0 \qquad (j = 1, \ldots, n) \quad . \quad . \tag{8} $$

by replacing L by any path connecting two zeros of the integrand, or by a loop similar to L but encircling only some (and not all) singularities of the integrand, or by a closed contour. Thus we get

$$ 2^{n+1} - 1 + \binom{n+1}{2} $$

different solutions.

Integral representations discerning the various solutions by different *paths* of integration in general occur when the system of linear partial differential equations dealt with does not have the greatest possible number of linearly independent solutions. (9), for instance, does not have the greatest possible number of linearly independent solutions of n equations of second order, viz. 2^n, because it implies the $\frac{1}{2}n(n-1)$ equations (*ibid.*, equation 7, 4):

$$(x_j - x_k)\frac{\partial^2 z}{\partial x_j \partial x_k} - \beta_k \frac{\partial z}{\partial x_j} + \beta_j \frac{\partial z}{\partial x_k} = 0 \qquad (j, k = 1, \ldots, n;\ j \neq k). \tag{9}$$

Sometimes mixed types of solutions occur; that is to say, the solution is of the type (2) in some of the variables, and of the type (7) in the other variables.

Integration of the System of Φ_2.

4. As an example, I shall investigate by the method described in section 3 the solutions of the system of two linear partial differential equations:

$$\left.\begin{aligned} x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x \partial y} + (\gamma - x)\frac{\partial z}{\partial x} - \beta z = 0, \\ x\frac{\partial^2 z}{\partial x \partial y} + y\frac{\partial^2 z}{\partial y^2} + (\gamma - y)\frac{\partial z}{\partial y} - \beta' z = 0. \end{aligned}\right\} \quad . \quad . \quad . \tag{10}$$

This system was also investigated by Horn (1931, pp. 394, 397 *et seq.*; 1935, pp. 640, 665 *et seq.*, 674; 1936, p. 289; 1938 *b*).

The only solution of (10) which can be expanded in ascending powers of x and y is Humbert's series (Humbert, 1920–21, p. 74):

$$\Phi_2(\beta, \beta';\ \gamma;\ x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\beta)_m (\beta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}. \quad . \quad . \tag{11}$$

Instead of the usual integral representation of Φ_2 (*ibid.*, p. 79),

$$\Phi_2(\beta, \beta';\ \gamma;\ x, y) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')} \iint u^{\beta-1} v^{\beta'-1} (1-u-v)^{\gamma-\beta-\beta'-1} e^{ux+vy}\, du\, dv \tag{12}$$

$$(u \geq 0,\ v \geq 0,\ 1-u-v \geq 0),$$

we start with the representation by a single integral (Erdélyi, 1937 *b*, equation 8, 6):

$$\Phi_2(\beta, \beta';\ \gamma;\ x, y) = \frac{\Gamma(\gamma)}{2\pi i} \int_{-\infty}^{(0+,\, x+,\, y+)} e^s s^{-\gamma} \left(1 - \frac{x}{s}\right)^{-\beta} \left(1 - \frac{y}{s}\right)^{-\beta'} ds, \quad . \tag{13}$$

in which

$$|\arg s| \leq \pi, \qquad |\arg (s-x)| \leq \pi, \qquad |\arg (s-y)| \leq \pi$$

is to be taken.

The connection between (12) and (13) is the same as between the dual integral representations of Kummer's series (3), which is the one-variable analogue to Φ_2. This connection can be elucidated by Laplace's transform (Erdélyi, 1937 *a*).

The integral representation (13) suggests that we shall try a solution of (10) of the form

$$z = c\int_a^b e^s s^{\beta+\beta'-\gamma}(s-x)^{-\beta}(s-y)^{-\beta'}ds, \tag{14}$$

in which a and b represent suitable limits of integration and c is an arbitrary constant.

Those solutions of (10) which can be represented by an integral of the type (14) are discussed in the subsequent section.

5. Supposing that (14) converges sufficiently strongly, we can differentiate under the sign of integration when substituting (14) in (10), and obtain

$$\left\{x\frac{\partial^2}{\partial x^2}+y\frac{\partial^2}{\partial x\partial y}+(\gamma-x)\frac{\partial}{\partial x}-\beta\right\}\int_a^b e^s s^{\beta+\beta'-\gamma}(s-x)^{-\beta}(s-y)^{-\beta'}ds$$
$$=\int_a^b\left\{\frac{\beta(\beta+1)}{(s-x)^2}x+\frac{\beta\beta' y}{(s-x)(s-y)}+\frac{\gamma-x}{s-x}\beta-\beta\right\}e^s s^{\beta+\beta'-\gamma}(s-x)^{-\beta}(s-y)^{-\beta'}ds.$$

But this is equal to

$$\beta\int_a^b\left(\frac{\beta+1}{s-x}+\frac{\beta'}{s-y}+\frac{\gamma-\beta-\beta'-1}{s}-1\right)e^s s^{\beta+\beta'-\gamma+1}(s-x)^{-\beta-1}(s-y)^{-\beta'}ds.$$

This last expression may be written

$$-\beta\int_a^b\frac{d}{ds}\{e^s s^{\beta+\beta'-\gamma+1}(s-x)^{-\beta-1}(s-y)^{-\beta'}\}ds.$$

Hence

$$\left\{x\frac{\partial^2}{\partial x^2}+y\frac{\partial^2}{\partial x\partial y}+(\gamma-x)\frac{\partial}{\partial x}-\beta\right\}\int_a^b e^s s^{\beta+\beta'-\gamma}(s-x)^{-\beta}(s-y)^{-\beta'}ds$$
$$=-\beta\Big[e^s s^{\beta+\beta'-\gamma+1}(s-x)^{-\beta-1}(s-y)^{-\beta'}\Big]_a^b,$$

and similarly

$$\left\{x\frac{\partial^2}{\partial x\partial y}+y\frac{\partial^2}{\partial y^2}+(\gamma-y)\frac{\partial}{\partial y}-\beta'\right\}\int_a^b e^s s^{\beta+\beta'-\gamma}(s-x)^{-\beta}(s-y)^{-\beta'}ds$$
$$=-\beta'\Big[e^s s^{\beta+\beta'-\gamma+1}(s-x)^{-\beta}(s-y)^{-\beta'-1}\Big]_a^b.$$

From this it is seen that (14) is a solution of (10) provided that the path of integration is either a closed contour on the Riemann surface of $e^s s^{\beta+\beta'-\gamma+1}(s-x)^{-\beta-1}(s-y)^{-\beta'-1}$ or a contour connecting two zeros of this function.

So far it is *not known* whether we have obtained the *general* solution of (10); that is to say, whether *every* solution of (10) is expressible in terms of integrals of the type (10). In section 8 it will be proved that (14) *does* represent the general solution.

Solutions in Convergent Series.

6. Discussing those solutions of (10) which are representable by integrals of the type (14), we begin with solutions representable by *definite* integrals (14). We obtain all possible solutions of this type by taking in (14) for the path of integration either a closed contour or a contour connecting two finite zeros of $s^{\beta+\beta'-\gamma+1}(s-x)^{-\beta-1}(s-y)^{-\beta'-1}$.

For the sake of simplicity let us assume throughout in this section

$$\mathbf{R}(\beta) < -1, \qquad \mathbf{R}(\beta') < -1, \qquad \mathbf{R}(\beta+\beta'-\gamma) > -1. \quad . \quad . \quad (15)$$

If these inequalities hold, every solution of the type described in the first paragraph of this section can be reduced to a linear combination of the three solutions:

$$z_1 = \frac{\Gamma(2+\beta'-\gamma)y^{-\beta'}}{\Gamma(1-\beta)\Gamma(\beta+\beta'-\gamma+1)}\int_0^x e^s s^{\beta+\beta'-\gamma}(x-s)^{-\beta}\left(1-\frac{s}{y}\right)^{-\beta'} ds, \quad . \quad (16)$$

$$z_2 = \frac{\Gamma(2+\beta-\gamma)x^{-\beta}}{\Gamma(1-\beta')\Gamma(\beta+\beta'-\gamma+1)}\int_0^y e^s s^{\beta+\beta'-\gamma}(y-s)^{-\beta'}\left(1-\frac{s}{x}\right)^{-\beta} ds, \quad . \quad (17)$$

$$z_3 = \frac{\Gamma(2-\beta-\beta')}{\Gamma(1-\beta)\Gamma(1-\beta')}\int_x^y e^s s^{\beta+\beta'-\gamma}(s-x)^{-\beta}(y-s)^{-\beta'} ds. \quad . \quad . \quad . \quad (18)$$

In addition to (15), in (16):

$$y \neq 0,\ y \neq x; \qquad \arg s = \arg(x-s) = \arg x; \qquad \left(1-\frac{s}{y}\right)^{-\beta'} \to 1 \quad \text{when} \quad s \to 0;$$

in (17):

$$x \neq 0,\ x \neq y; \qquad \arg s = \arg(y-s) = \arg y; \qquad \left(1-\frac{s}{x}\right)^{-\beta} \to 1 \quad \text{when} \quad s \to 0;$$

and in (18):

$$x \neq 0,\ y \neq 0; \ \arg(s-x) = \arg(y-s) = \arg(y-s); \ s^{\beta+\beta'-\gamma} \to x^{\beta+\beta'-\gamma} \text{ when } s \to x$$

is supposed to be taken.

Unfortunately the solutions z_1, z_2 and z_3 are *not linearly independent* of each other, and therefore they cannot constitute a fundamental system of solutions of the system (10).

To prove this, we remark that

$$\int_C e^s s^{\beta+\beta'-\gamma}(s-x)^{-\beta}(s-y)^{-\beta'}ds=0, \qquad (19)$$

when C is any closed contour containing no singular point of the integrand inside. Now suppose that the points $s=0$, $s=x$, $s=y$ of the complex s-plane are not collinear. Let C be a closed contour entirely inside the triangle with vertices $s=0$, x, y. Then (19) holds. Now let C expand till it coincides with the circumference of the triangle 0, x, y. Then we have, in the limit, with appropriate values of arg s, arg $(s-x)$ and arg $(s-y)$ the relation

$$\left(\int_0^x+\int_x^y+\int_y^0\right)e^s s^{\beta+\beta'-\gamma}(s-x)^{-\beta}(s-y)^{-\beta'}ds=0, \qquad (20)$$

and this is a linear relation between z_1, z_2, z_3.

On the other hand, this is the only linear relation connecting z_1, z_2 and z_3. For supposing that a second linear and homogeneous relation between our three solutions, independent of (20), could exist, we should come to a contradiction. This second relation is either a linear homogeneous relation between z_1 and z_2 only, or else, if it contains z_3, it can be reduced by means of (20) to a linear homogeneous relation between z_1 and z_2 only, say

$$c_1z_1+c_2z_2=0.$$

But no relation of this kind with non-vanishing coefficients can exist, because z_1 vanishes when $x\to 0$, $y\neq 0$ and z_2 does not vanish. Therefore z_1 and z_2 are linearly independent of each other.

The restrictions (15) are by no means essential. They can be removed in the usual way (*e.g.* Whittaker-Watson, 1927, § 12.43) by introducing double-loop integrals into (16)–(18).

7. The three solutions of (10) obtained in the preceding section can be expressed in terms of Humbert's confluent hypergeometric function of two variables (Humbert, 1920–21, p. 74):

$$\Phi_1(a,\beta,\gamma;\ x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(\beta)_m}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}. \qquad (|x|<1). \qquad (21)$$

The connection between Φ_1 and our solutions z_1, z_2, z_3 is understood from the integral representation given by Humbert (*ibid.*, p. 79):

$$\Phi_1(a,\beta,\gamma;\ x,y)=\frac{\Gamma(\gamma)}{\Gamma(a)\Gamma(\gamma-a)}\int_0^1 u^{a-1}(1-u)^{\gamma-a-1}(1-xu)^{-\beta}e^{uy}du \qquad (22)$$

$$[\mathbf{R}(\gamma)>\mathbf{R}(a)>0;\quad |\arg(1-x)|<\pi,\quad x\neq 1].$$

This integral representation shows that Φ_1 behaves in x like an ordinary hypergeometric series, in y like a confluent hypergeometric series. This behaviour is also seen from the transformation formula

$$\Phi_1(\alpha, \beta, \gamma;\ x, y) = (1-x)^{-\beta} e^{y} \Phi_1\left(\gamma-\alpha, \beta, \gamma;\ \frac{x}{x-1}, -y\right), \qquad (23)$$

which follows from (22) by replacing u by $1-v$. This transformation formula is in x identical with Euler's transformation of the ordinary hypergeometric series, in y identical with Kummer's transformation of the confluent hypergeometric series. (23) shows that every integral of the type (22) can be expressed in two different ways by Humbert's series Φ_1.

From (22) and (23) it follows that z_1, z_2 and z_3, as defined by (16)–(18), can be expressed, each in two different ways, in terms of Humbert's series Φ_1. A simple algebraic calculation yields

$$z_1 = x^{\beta'-\gamma+1} y^{-\beta'} \Phi_1\left(\beta+\beta'-\gamma+1, \beta', \beta'-\gamma+2;\ \frac{x}{y}, x\right) \qquad (|x|<|y|)$$

$$= x^{\beta'-\gamma+1} (y-x)^{-\beta'} e^{x} \Phi_1\left(1-\beta, \beta', \beta'-\gamma+2;\ \frac{x}{x-y}, -x\right) \quad (|x|<|x-y|), \quad (24)$$

$$z_2 = x^{-\beta} y^{\beta-\gamma+1} \Phi_1\left(\beta+\beta'-\gamma+1, \beta, \beta-\gamma+2;\ \frac{y}{x}, y\right) \qquad (|y|<|x|)$$

$$= (x-y)^{-\beta} y^{\beta-\gamma+1} e^{y} \Phi_1\left(1-\beta', \beta, \beta-\gamma+2;\ \frac{y}{y-x}, -y\right) \quad (|y|<|y-x|), \quad (25)$$

$$z_3 = x^{\beta+\beta'-\gamma} (y-x)^{1-\beta-\beta'} e^{x} \Phi_1\left(1-\beta, \gamma-\beta-\beta', 2-\beta-\beta';\ 1-\frac{y}{x}, y-x\right)$$
$$(|x|>|x-y|)$$

$$= y^{\beta+\beta'-\gamma} (y-x)^{1-\beta-\beta'} e^{y} \Phi_1\left(1-\beta', \gamma-\beta-\beta', 2-\beta-\beta';\ 1-\frac{x}{y}, x-y\right)$$
$$(|y|>|y-x|). \quad (26)$$

From this it is seen (i) that the solutions of (10) obtained in the preceding section are expressible in terms of hypergeometric functions of two variables, convergent in certain tubes of the four-dimensional space of x and y, and (ii) that (10) is equivalent to, and in six different ways transformable into, the system of linear partial differential equations * which belongs to Φ_1. The six different ways of transformation of (10) into a

* This system was given by Humbert (1920–21), p. 77, and also studied by Horn (1935), p. 643.

system of the type of the system of Φ_1 correspond to certain transformations of the Φ_1-system into another Φ_1-system with different parameters.

Thus our method also furnishes us with some information about the *connection* between *different* systems of linear partial differential equations of hypergeometric type.

Moreover, we learn some new transformation formulæ of Φ_1. For using, for instance, the first of each pair of equations (24)–(26) we obtain from (20):

$$\begin{aligned}\frac{x^{\beta+\beta'-\gamma}(y-x)^{1-\beta-\beta'}e^{x}}{\Gamma(2-\beta-\beta')\Gamma(\beta+\beta'-\gamma+1)}e^{\pi i\beta}\Phi_1\left(1-\beta', \gamma-\beta-\beta', 2-\beta-\beta';\ 1-\frac{y}{x}, y-x\right)\\ =\frac{x^{-\beta}y^{\beta-\gamma+1}}{\Gamma(\beta-\gamma+2)}\Phi_1\left(\beta+\beta'-\gamma+1, \beta, \beta-\gamma+2;\ \frac{y}{x}, y\right)\\ -\frac{x^{\beta'-\gamma+1}y^{-\beta'}}{\Gamma(\beta'-\gamma+2)}\Phi_1\left(\beta+\beta'-\gamma+1, \beta', \beta'-\gamma+2;\ \frac{x}{y}, x\right). \quad (27)\end{aligned}$$

Some other transformation formulæ of a similar kind, obtainable from (27) by application of the transformation (23) to some or all terms in (27), may be left to the reader.

8. So far as I see the only solution of (10) beside z_1, z_2, z_3 expressible by *convergent* hypergeometric series of two variables is the well-known solution

$$z_0=\Phi_2(\beta, \beta';\ \gamma;\ x, y). \quad . \quad . \quad . \quad . \quad . \quad (28)$$

This solution is obtained from (14) by taking a contour beginning at $-\infty$, encircling all three points $s=0$, x, y in positive direction and returning to $-\infty$, and by taking an appropriate value of the constant factor c in (14). Applying the generalisation of Kummer's transformation, which was discussed by the present writer (Erdélyi, 1937 *b*, § 9) and independently rediscovered by Horn (Horn, 1938 *b*, p. 189), to Φ_2, we have three different representations of z_0, namely:

$$\begin{aligned}z_0&=\Phi_2(\beta, \beta';\ \gamma;\ x, y),\\ &=e^{x}\Phi_2(\gamma-\beta-\beta', \beta';\ \gamma;\ -x, y-x), \quad . \quad . \quad . \quad (29)\\ &=e^{y}\Phi_2(\beta, \gamma-\beta-\beta';\ \gamma;\ x-y, -y).\end{aligned}$$

It is rather important to remark that in general, *i.e.* save for certain exceptional values of the parameters, z_0, z_1 and z_2 are linearly independent. For if a linear relation of the form

$$c_0z_0+c_1z_1+c_2z_2=0 \quad . \quad . \quad . \quad . \quad . \quad (30)$$

could exist, it would be valid for $x\to 0$, $y\to 0$. But for $x\to 0, y\to 0$ with the assumptions (15) we get from (24), (25), (28) and (30) that $c_0=0$

and (30) would reduce to

$$c_1 z_1 + c_2 z_2 = 0,$$

from which, we have seen in section 6, $c_1 = c_2 = 0$ follow.

On the other hand, as it is known (*e.g.* Humbert, 1920–21, p. 78) that every solution of (10) is expressible as a linear combination of *three* linearly independent solutions, we see that *every solution of* (10) *is expressible in the form*

$$z = c_0 z_0 + c_1 z_1 + c_2 z_2.$$

So far as I know z_0, z_1 and z_2 is the first set of solutions of (10) in terms of hypergeometric series of two variables.

Now it is possible to answer the question whether every solution of (10) is expressible in terms of integrals of the type (14), which was raised in section 5, in the affirmative.

Thus we have obtained in sections 6–8 nine convergent hypergeometric series of two variables representing four different solutions of (10), three of which are linearly independent. The solutions obtained thus are valid without regard to the preliminary restrictions (15) which were introduced for the sake of simplicity only.

Solutions in Divergent Series.

9. After having enumerated all solutions of (10) which are representable by *convergent* hypergeometric series of two variables, we proceed by obtaining another fundamental set of solutions which can be represented *asymptotically* by *divergent* hypergeometric series of two variables.

The meaning of the solutions to be obtained in this section will be learned by comparing (10) with Whittaker's confluent hypergeometric differential equation (Whittaker-Watson, 1927, chap. xvi):

$$\frac{d^2W}{dx^2} + \left(-\frac{1}{4} + \frac{k}{x} + \frac{m^2 - \frac{1}{4}}{x^2} \right) W = 0.$$

The solutions of (10) obtained hitherto correspond to the solution of Whittaker's equation in terms of $M_{k,m}$-functions; the solutions which will be obtained in this section correspond to the solutions of Whittaker's equation in terms of $W_{k,m}$-functions.

Suppose for a moment that the restrictions (15) for the parameters hold. Then $e^s s^{\beta+\beta'-\gamma+1}(s-x)^{-\beta-1}(s-y)^{-\beta'-1}$ has three finite zeros, namely, the points $s=0$, x, y. It is therefore possible to obtain three solutions of (10) by taking in (14) the path of integration to be a straight line (say) starting at one of the zeros and going to $-\infty$. Introducing

$t = -s$ as a new variable of integration and adjusting the constant c of (14) properly, we have the three solutions of (10):

$$z_4 = \frac{x^{-\beta}y^{-\beta'}}{\Gamma(\beta+\beta'-\gamma+1)}\int_0^\infty e^{-t}t^{\beta+\beta'-\gamma}\left(1+\frac{t}{x}\right)^{-\beta}\left(1+\frac{t}{y}\right)^{-\beta'}dt \quad . \quad . \quad . \quad . \quad (31)$$

$$[\mathbf{R}(\beta+\beta'-\gamma) > -1;\ x \neq 0, y \neq 0;\ |\arg x| < \pi, |\arg y| < \pi],$$

$$z_5 = \frac{(-x)^{\beta+\beta'-\gamma}(y-x)^{-\beta'}}{\Gamma(1-\beta)}\int_{-x}^\infty e^{-t}(t+x)^{-\beta}\left(1+\frac{t+x}{y-x}\right)^{-\beta'}\left(1+\frac{t+x}{-x}\right)^{\beta+\beta'-\gamma}dt \quad . \quad (32)$$

$$[\mathbf{R}(\beta) < 1;\ x \neq 0, y \neq x;\ |\arg(-x)| < \pi, |\arg(y-x)| < \pi],$$

$$z_6 = \frac{(x-y)^{-\beta}(-y)^{\beta+\beta'-\gamma}}{\Gamma(1-\beta')}\int_{-y}^\infty e^{-t}(t+y)^{-\beta'}\left(1+\frac{t+y}{x-y}\right)^{-\beta}\left(1+\frac{t+y}{-y}\right)^{\beta+\beta'-\gamma}dt \quad . \quad (33)$$

$$[\mathbf{R}(\beta') < 1;\ y \neq 0, x \neq y;\ |\arg(x-y)| < \pi, |\arg(-y)| < \pi].$$

The principal values of $t^{\beta+\beta'-\gamma}$, $(t+x)^{-\beta}$, $(t+y)^{-\beta'}$ are to be taken. All other powers like $(1+t/x)^{-\beta}, \ldots, (1+(t+y)/(-y))^{\beta+\beta'-\gamma}$ are uniquely determined by fixing their values to be 1 on the lower limit of integration.

Using loop integrals instead of those of (31)–(33), the following integral representations of z_4, z_5 and z_6 can be derived:—

$$z_4 = x^{-\beta}y^{-\beta'}\frac{\Gamma(\gamma-\beta-\beta')}{2\pi i}\int_{-\infty}^{(0+)} e^s s^{\beta+\beta'-\gamma}\left(1-\frac{s}{x}\right)^{-\beta}\left(1-\frac{s}{y}\right)^{-\beta'}ds \quad . \quad . \quad (34)$$

$$[\beta+\beta'-\gamma \neq 0, 1, 2, \ldots;\ x \neq 0, y \neq 0;\ |\arg s| \leq \pi, |\arg x| < \pi, |\arg y| < \pi],$$

$$z_5 = (-x)^{\beta+\beta'-\gamma}(y-x)^{-\beta'}\frac{\Gamma(\beta)}{2\pi i}\int_{-\infty}^{(x+)} e^s(s-x)^{-\beta}\left(1-\frac{s-x}{y-x}\right)^{-\beta'}\left(1-\frac{s-x}{-x}\right)^{\beta+\beta'-\gamma}ds \ . \ (35)$$

$$[\beta \neq 0, -1, -2, \ldots;\ x \neq 0, y \neq x;\ |\arg(s-x)| \leq \pi, |\arg(-x)| < \pi, |\arg(y-x)| < \pi],$$

$$z_6 = (x-y)^{-\beta}(-y)^{\beta+\beta'-\gamma}\frac{\Gamma(\beta')}{2\pi i}\int_{-\infty}^{(y+)} e^s(s-y)^{-\beta'}\left(1-\frac{s-y}{x-y}\right)^{-\beta}\left(1-\frac{s-y}{-y}\right)^{\beta+\beta'-\gamma}ds \ . \ (36)$$

$$[\beta' \neq 0, -1, -2, \ldots;\ y \neq 0, x \neq y;\ |\arg(s-y)| \leq \pi, |\arg(-y)| < \pi, |\arg(x-y)| < \pi].$$

The loops are understood not to include any singular point of the integrand other than that mentioned on the top of the integral sign. The powers $(1-s/y)^{-\beta'}, \ldots, (1-(s-y)/(-y))^{\beta+\beta'-\gamma}$ are uniquely determined by the convention that they reduce to 1 when s approaches the singular point inside the loop.

The integral representations (31)–(36) define z_4, z_5 and z_6 for all values of the parameters. By turning round the path of integration wider ranges of the variables can be admitted.

10. The three solutions of (10) introduced in the preceding section

can be represented asymptotically by ***divergent*** hypergeometric series of the type

$$\Phi_4\left(\alpha, \beta, \beta'; -\frac{1}{x}, -\frac{1}{y}\right) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{m!n!(-x)^m(-y)^n}. \quad . \quad . \quad (37)$$

It is easily seen that this series is divergent for every pair of finite values of x and y save for zero or negative integer values of α or both β and β'. In these exceptional cases of terminating series Φ_4 can be transformed into a terminating Φ_2.

Φ_4 is the generalisation of the (when not terminating) divergent hypergeometric series

$${}_2F_0\left(\alpha, \beta; -\frac{1}{x}\right) = \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m}{m!(-x)^m}$$

to two variables, just as Φ_2 is the extension of ${}_1F_1$ to two variables (see section 4). Knowing that the asymptotic representation of the solutions of (4) for large values of w yields series of the type ${}_2F_0$, it seems probable that the asymptotic representations of solutions of (10) for large values of the variables contain series of the type Φ_4. It will be seen that the three solutions introduced in the preceding section possess quite simple asymptotic representations, only one series of the type Φ_4 occurring in each of the asymptotic representations of z_4, z_5 and z_6.

(37) does *not* define a definite function, because there is an infinite number of functions having the same asymptotic expansion. In order to define Φ_4 *uniquely* we may use either of the integral representations:

$$\Phi_4\left(\alpha, \beta, \beta'; -\frac{1}{x}, -\frac{1}{y}\right) = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} e^{-t}t^{\alpha-1}\left(1+\frac{t}{x}\right)^{-\beta}\left(1+\frac{t}{y}\right)^{-\beta'} dt,$$

$$= \frac{\Gamma(1-\alpha)}{2\pi i}\int_{-\infty}^{(0+)} e^{s}s^{\alpha-1}\left(1-\frac{s}{x}\right)^{-\beta}\left(1-\frac{s}{y}\right)^{-\beta'} ds. \quad . \quad (38)$$

In the first integral $\mathbf{R}(\alpha) > 0$ has to be supposed and the principal value of $t^{\alpha-1}$ is to be taken; in the second representation $\alpha \neq 1, 2, 3, \ldots$ is assumed and $|\arg s| \leq \pi$ is to be taken; in both representations

$$x \neq 0, \quad y \neq 0, \quad |\arg x| < \pi, \quad |\arg y| < \pi,$$

and those branches of the powers $\left(1+\frac{t}{x}\right)^{-\beta}, \ldots, \left(1-\frac{s}{y}\right)^{-\beta'}$ may be taken which reduce to 1 when t or s approaches 0.

It is quite clear that both integral representations (38) are representations of the *same* function valid in different regions of the parameters (see, *e.g.*, Whittaker-Watson, 1927, § 12.22). Moreover, (38) is in

agreement with (37), for computing the asymptotic representation of Φ_4 from (38) by Watson's Lemma (see, for instance, Watson, 1922, p. 236) when $|x| \to \infty$, $|y| \to \infty$, $x/y=$cons. we obtain, for instance, from the first integral (38)

$$\Phi_4\left(\alpha, \beta, \beta';\ -\frac{1}{x}, -\frac{1}{y}\right)=\frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-t}t^{\alpha-1}\sum_{m=0}^{\infty}\frac{(\beta)_m}{m!}\left(-\frac{t}{x}\right)^m\sum_{n=0}^{\infty}\frac{(\beta')_n}{n!}\left(-\frac{t}{y}\right)^n dt$$

$$\sim\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\beta)_m(\beta')_n}{m!n!}\left(-\frac{1}{x}\right)^m\left(-\frac{1}{y}\right)^n\frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-t}t^{\alpha+m+n-1}dt,$$

and this is equivalent to (37).

Now comparing (38) with (31)–(33) or with (34)–(36), it is seen immediately that

$$z_4=x^{-\beta}y^{-\beta'}\Phi_4\left(\beta+\beta'-\gamma+1, \beta, \beta';\ -\frac{1}{x}, -\frac{1}{y}\right), \qquad (39)$$

$$z_5=(-x)^{\beta+\beta'-\gamma}(y-x)^{-\beta'}e^x\Phi_4\left(1-\beta, \gamma-\beta-\beta', \beta';\ \frac{1}{x}, \frac{1}{x-y}\right) \qquad (40)$$

and

$$z_6=(x-y)^{-\beta}(-y)^{\beta+\beta'-\gamma}e^y\Phi_4\left(1-\beta', \beta, \gamma-\beta-\beta', \frac{1}{y-x}, \frac{1}{y}\right). \qquad (41)$$

It follows from the different asymptotic behaviour of z_4, z_5 and z_6 for large values of $|x|$, $|y|$ and $|x-y|$ that these three solutions are linearly independent. Thus they form a fundamental system of solutions of (10).

The solutions z_4, z_5 and z_6 have been obtained recently by Horn (Horn, 1938 *b*) by a different method. Horn's integral representation is, however, different from, and more complicated than, our integral representations. Horn's integral representation as well as more general integral representations of our solutions can be derived from those given in this paper by fractional integration by parts.*

Solutions in Mixed Series.

11. In conclusion a third and last triplet of solutions of (10) may be obtained. The hypergeometric series connected with this set of solutions are of a *mixed* type, being convergent in one of the variables and divergent (and asymptotic) in the other.

Solutions of this type are obtained by taking a path of integration,

* I have dealt with the application of fractional integration by parts to integral representations of hypergeometric functions in some forthcoming papers (the last three in the list of references).

which is a loop encircling two of the three singular points of the integrand of (14). The third singular point is understood to be outside the loop.

Choosing the constant c in (14) properly we obtain the integrals

$$z_7 = \frac{\Gamma(\gamma-\beta')}{2\pi i}\int_{-\infty}^{(0+,x+)} e^s s^{\beta+\beta'-\gamma}(s-x)^{-\beta}(y-s)^{-\beta'}ds \quad . \quad . \quad (42)$$

$[\gamma-\beta'\neq 0, -1, -2, \ldots; x\neq y, y\neq 0; |\arg s|\leq\pi, |\arg(s-x)|\leq\pi,$
$\arg(y-s)\to 0$ when $s\to-\infty$],

$$z_8 = \frac{\Gamma(\gamma-\beta)}{2\pi i}\int_{-\infty}^{(0+,y+)} e^s s^{\beta+\beta'-\gamma}(x-s)^{-\beta}(s-y)^{-\beta'}ds \quad . \quad . \quad (43)$$

$[\gamma-\beta\neq 0, -1, -2, \ldots; x\neq 0, y\neq x; |\arg s|\leq\pi, |\arg(s-y)|\leq\pi,$
$\arg(x-s)\to 0$ when $s\to\infty$],

$$z_9 = \frac{\Gamma(\beta+\beta')}{2\pi i}\int_{-\infty}^{(x+,y+)} e^s(-s)^{\beta+\beta'-\gamma}(s-x)^{-\beta}(s-y)^{-\beta'}ds. \quad . \quad (44)$$

$[\beta+\beta'\neq 0, -1, -2, \ldots; x\neq 0, y\neq 0; |\arg(s-x)|\leq\pi, |\arg(s-y)|\leq\pi,$
$\arg(-s)\to 0$ when $s\to-\infty$].

The three solutions obtained thus are expressible in terms of the hypergeometric function

$$\Phi_8\left(\beta, \beta', \gamma;\ x, \frac{1}{y}\right) = \frac{\Gamma(\gamma)}{2\pi i}\int_{-\infty}^{(0+,x+)} e^s s^{\beta-\gamma}(s-x)^{-\beta}\left(1-\frac{s}{y}\right)^{-\beta'}ds \quad . \quad (45)$$

$\Big[\gamma\neq 0, -1, -2, \ldots; y\neq 0, x\neq y; |\arg s|\leq\pi\ |\arg(s-x)|\leq\pi,$
$\arg\left(1-\frac{s}{y}\right)\to 0$ when $s\to 0\Big]$.

Obviously we have

$$\Phi_8(\beta, \beta', \gamma;\ 0, 0) = 1. \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Replacing s in (45) by $\sigma+x$ we obtain

$$\Phi_8\left(\beta, \beta', \gamma;\ x, \frac{1}{y}\right) = e^x\frac{\Gamma(\gamma)}{2\pi i}\int_{-\infty}^{(0+,-x+)} e^\sigma\sigma^{-\beta}(\sigma+x)^{\beta-\gamma}\left(1-\frac{\sigma+x}{y}\right)^{-\beta'}d\sigma$$

$$= \left(1-\frac{x}{y}\right)^{-\beta'} e^x\frac{\Gamma(\gamma)}{2\pi i}\int_{-\infty}^{(0+,-x+)} e^\sigma\sigma^{-\beta}(\sigma+x)^{\beta-\gamma}\left(1-\frac{\sigma}{y-x}\right)^{-\beta'}d\sigma.$$

Using on the right of the last equation (45), we arrive at the transformation formula:

$$\Phi_8\left(\beta, \beta', \gamma;\ x, \frac{1}{y}\right) = \left(1-\frac{x}{y}\right)^{-\beta'} e^x\Phi_8\left(\gamma-\beta, \beta', \gamma;\ -x, \frac{1}{y-x}\right). \quad . \quad (47)$$

which enables us to express each of the integrals (42)–(44) in *two* different ways in terms of the Φ_5-function.

Doing so we obtain the following representations of our third triplet of solutions of (10):—

$$z_7 = y^{-\beta'}\Phi_5\left(\beta, \beta', \gamma-\beta'; \ x, \frac{1}{y}\right)$$

$$= (y-x)^{-\beta'}e^{x}\Phi_5\left(\gamma-\beta-\beta', \beta', \gamma-\beta'; \ -x, \frac{1}{y-x}\right), \quad . \quad . \quad (48)$$

$$z_8 = x^{-\beta}\Phi_5\left(\beta', \beta, \gamma-\beta; \ y, \frac{1}{x}\right)$$

$$= (x-y)^{-\beta}e^{y}\Phi_5\left(\gamma-\beta-\beta', \beta, \gamma-\beta; \ -y, \frac{1}{x-y}\right), \quad . \quad . \quad (49)$$

$$z_9 = (-x)^{\beta+\beta'-\gamma}e^{x}\Phi_5\left(\beta', \gamma-\beta-\beta', \beta+\beta'; \ y-x, \ -\frac{1}{x}\right)$$

$$= (-y)^{\beta+\beta'-\gamma}e^{y}\Phi_5\left(\beta, \gamma-\beta-\beta', \beta+\beta'; \ x-y, \ -\frac{1}{y}\right). \quad . \quad . \quad (50)$$

In connection with these solutions certain expansions of Φ_5 may be of some importance.

If $|y| > |x|$, it is possible to take in (45) a path of integration entirely outside of the circle $|s| = |x|$. Along this path of integration the expansion

$$(s-x)^{-\beta} = s^{-\beta}\left(1-\frac{x}{s}\right)^{-\beta} = s^{-\beta}\sum_{m=0}^{\infty}\frac{(\beta)_m}{m!}\left(\frac{x}{s}\right)^m$$

converges absolutely and uniformly, and hence term by term integration is permissible. Thus we obtain from (45):

$$\Phi_5\left(\beta, \beta', \gamma; \ x, \frac{1}{y}\right) = \frac{\Gamma(\gamma)}{2\pi i}\int_{-\infty}^{(0+, x+)} e^{s}s^{-\gamma}\sum_{m=0}^{\infty}\frac{(\beta)_m}{m!}\left(\frac{x}{s}\right)^m\left(1-\frac{s}{y}\right)^{-\beta'}ds$$

$$= \sum_{m=0}^{\infty}\frac{(\beta)_m}{m!}x^m\frac{\Gamma(\gamma)}{2\pi i}\int_{-\infty}^{(0+)} e^{s}s^{-\gamma-m}\left(1-\frac{s}{y}\right)^{-\beta'}ds. \quad . \quad (51)$$

The integrand of the last integral has no singularity at $s=x$, and hence the path of integration encircles only the singularity $s=0$; $s=y$ lies outside of the loop of integration.

In agreement with a former paper * I define the divergent hypergeometric series,

$${}_2F_0(\alpha, \beta; \ x) = \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{n!}x^n, \quad . \quad . \quad . \quad . \quad . \quad (52)$$

* Erdélyi (in press, *Quart. Journ. Math.*, Oxford), section 10. The definition given in the paper quoted is in its shape different from, but equivalent with, the present definition.

uniquely by

$$ {}_2F_0\left(\alpha, \beta;\ -\frac{1}{x}\right) = \frac{\Gamma(1-\alpha)}{2\pi i}\int_{-\infty}^{(0+)} e^s s^{\alpha-1}\left(1-\frac{s}{x}\right)^{-\beta} ds \quad . \quad . \quad (53) $$

$$ \left[\alpha \neq 1, 2, 3, \ldots;\ x \neq 0;\ |\arg x| < \pi;\ |\arg s| \leq \pi, \arg\left(1-\frac{s}{x}\right) \to 0 \text{ when } s \to 0\right]. $$

With this notation we obtain from (51) the expansion of Φ_5, convergent provided that $|x| < |y|$:

$$ \Phi_5\left(\beta, \beta', \gamma;\ x, \frac{1}{y}\right) = \sum_{m=0}^{\infty} \frac{(\beta)_m}{(\gamma)_m}\frac{x^m}{m!}\,{}_2F_0\left(1-\gamma-m, \beta', -\frac{1}{y}\right), \quad . \quad . \quad (54) $$

which is an expansion in a series of Whittaker's confluent hypergeometric functions $W_{k,m}(y)$. A second expansion of a similar type of the same series follows from the transformation formula (47).

Replacing ${}_2F_0$ in (54) by the formal series (52) we obtain for Φ_5 the *formal expansion*

$$ \Phi_5\left(\beta, \beta', \gamma;\ x, \frac{1}{y}\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\beta)_m(\beta')_n}{(\gamma)_{m-n}}\frac{x^m}{m!}\frac{y^{-n}}{n!}, \quad . \quad . \quad (55) $$

which is of the character of a series convergent in x but divergent in y. This series represents Φ_5 asymptotically for small values of x and large values of y.

Thus we have for each of the solutions z_7, z_8 and z_9 two formal series representing the solutions referred to asymptotically in certain regions of the complex variables x and y.

SUMMARY.

12. Introductorily a brief account is given on the methods generally adopted for the integration of systems of linear partial differential equations of the hypergeometric type. In the main part of the paper the integration of the system of two linear partial differential equations of second order, of which Humbert's confluent hypergeometric series Φ_2 of two variables is one solution, is dealt with in some detail.

At first nine convergent hypergeometric series of two variables are obtained representing four different solutions of the system dealt with. Three of these solutions are linearly independent and constitute a fundamental system of solutions.

A second set of three linearly independent solutions is obtained and the asymptotic representation of the functions of this set by divergent hypergeometric series of two variables is discussed.

A third set of three solutions, each of which is asymptotically repre-

sentable by two hypergeometric series in two variables, convergent in one of the variables and divergent in the other, is given. Each of these solutions is also representable by a convergent series of Whittaker's confluent hypergeometric functions $W_{k,m}$.

Thus eighteen formal solutions of the system of two linear partial differential equations of second order have been obtained, representing ten different solutions of this system. Three sets of fundamental solutions have been discussed.

The method used enables us to find the full transformation scheme of the solutions introduced and to derive the monodromic group of the system of differential equations dealt with. The discussion of these points, however, is left to a subsequent paper.

REFERENCES TO LITERATURE.

APPELL, P., 1925. "Sur les fonctions hypergéométriques de plusieurs variables, les polynomes d'Hermite et autres fonctions sphériques dans l'hyperespaçe," *Mém. sci. math.*, Fasc. 3, 75 p.

APPELL, P., et KAMPÉ DE FÉRIET, J., 1926. *Fonctions hypergéométriques et hypersphériques. Polynoms d'Hermite*, Paris.

BORNGÄSSER, L., 1933. "Ueber hypergeometrische Funktionen zweier Veränderlichen," *Dissertation, Darmstadt.*

ERDÉLYI, A., 1936. "Ueber einige bestimmte Integrale, in denen die Whittakerschen $M_{k,m}$-Funktionen auftreten," *Math. Zeits.*, vol. xl, pp. 693–702.

—— 1937 *a*. "Der Zusammenhang zwischen verschiedenen Integraldarstellungen hypergeometrischer Funktionen," *Quart. Journ. Math.* (Oxford), vol. viii, pp. 200–213.

—— 1937 *b*. "Beitrag zur Theorie der konfluenten hypergeometrischen Funktionen von mehreren Veränderlichen," *S. Akad. Wiss. Wien*, vol. cxlvi, pp. 431–467.

—— 1939 *a*. "Integraldarstellungen für Produkte Whittakerscher Funktionen," *Nieuw Arch. Wisk.*, vol. xx, pp. 1–34.

—— 1939 *b*. "Note on the transformation of Eulerian hypergeometric integrals," *Quart. Journ. Math.*, Oxford, vol. x, pp. 129–134.

—— (In press). "Transformation of hypergeometric integrals by means of fractional integration by parts," *Quart. Journ. Math.*, Oxford.

—— (In press). Integral representations for Whittaker functions.

HORN, J., 1889. "Ueber die Convergenz der hypergeometrischen Reihen zweier und dreier Veränderlichen," *Math. Ann.*, vol. xxxiv, pp. 544–600.

—— 1931. "Hypergeometrische Funktionen zweier Veränderlichen," *Math. Ann.*, vol. cv, pp. 381–407.

HORN, J., 1935. "Hypergeometrische Funktionen zweier Veränderlichen," *Math. Ann.*, vol. cxi, pp. 638–677.

—— 1936. "Hypergeometrische Funktionen zweier Veränderlichen. Fortsetzung der Arbeit in Bd. 111," *Math. Ann.*, vol. cxiii, pp. 242–291.

—— 1938 *a*. "Hypergeometrische Funktionen zweier Veränderlichen im Schnittpunkt dreier Singularitäten," *Math. Ann.*, vol. cxv, pp. 435–455.

—— 1938 *b*. "Ueber eine hypergeometrische Funktion zweier Veränderlichen," *Monatshefte Math. Phys.*, vol. xlvii, pp. 186–194.

HUMBERT, P., 1920–21. "The confluent hypergeometric functions of two variables," *Proc. Roy. Soc. Edin.*, vol. xli, pp. 73–96.

LAURICELLA, G., 1893. "Sulle funzioni ipergeometriche a piu variabili," *Rend. Circ. mat. Palermo*, vol. vii, pp. 111–158.

WATSON, G. N., 1922. *A treatise on the theory of Bessel functions*, Cambridge.

WHITTAKER, E. T., 1931. "On the solution of differential equations by definite integrals," *Proc. Edin. Math. Soc.* (2), vol. ii, pp. 189–204.

WHITTAKER, E. T., and WATSON, G. N., 1927. *A course of modern analysis*, 4th ed., Cambridge.

(*Issued separately September* 28, 1939.)

XXIII.—Genetic Algebras. By **I. M. H. Etherington**, B.A.(Oxon), Ph.D.(Edin.), Mathematical Institute, University of Edinburgh.

(MS. received May 8, 1939. Revised MS. received September 5, 1939. Read July 3, 1939.)

CONTENTS.

§ 1. Introduction.

Two classes of linear algebras, generally non-associative, are defined in § 3 (*baric algebras*) and § 4 (*train algebras*), and the process of *duplication* of a linear algebra in § 5. These concepts, which will be discussed more fully elsewhere, arise naturally in the symbolism of genetics, as shown in §§ 6–15. Many of their properties express facts well known in genetics; and the processes of calculation which are fundamental in many problems of population genetics can be expressed as manipulations in the genetic algebras. In cases where inheritance is of a simple type (*e.g.* §§ 10–13, 15) this constitutes a new point of view, but perhaps amounts to little more than a change of notation as compared with existing methods. § 14, however, indicates the possibility of generalisations which would seem to be impossible by ordinary methods.

The occurrence of the genetic algebras may be described in general terms as follows. The mechanism of chromosome inheritance, in so far as it determines the probability distributions of genetic types in families and filial generations, and expresses itself through their frequency distributions, may be represented conveniently by algebraic symbols. Such a symbolism is described, for instance, by Jennings (1935, chap. ix);

many applications are given by Geppert and Koller (1938). It is shown in the present paper that the symbolism is equivalent to the use of a system of related linear algebras, in which multiplication (equivalent to the procedure of "chessboard diagrams") is commutative ($PQ = QP$) but non-associative ($PQ \cdot R \neq P \cdot QR$). A population (*i.e.* a distribution of genetic types) is represented by a normalised hypercomplex number in one or other algebra, according to the point of view from which it is specified. If P, Q are populations, the filial generation $P \times Q$ (*i.e.* the statistical population of offspring resulting from the random mating of individuals of P with individuals of Q) is obtained by multiplying two corresponding representations of P and Q; and from this requirement of the symbolism it will be obvious why multiplication must be non-associative. It must be understood that a population may mean a single individual, or rather the information which we may have concerning him in the form of a probability distribution.

Inheritance will be called *symmetrical* if the sex of a parent does not affect the distribution of gametic types produced. Paying attention only to the inheritance of gene differences (not of phenotypes), every regular mode of symmetrical inheritance in theoretical genetics has its fundamental *gametic algebra*, from which other algebras (*zygotic*, etc.) are deduced by duplication. From the nature of the symbolism these are of necessity baric algebras; but it appears on closer examination that they belong in all cases to the narrower category of train algebras.

(The fundamental algebras can be modified to take account of various kinds of selection. They are then no longer train algebras, although the baric property and the relation of duplication sometimes persist.)

Symmetry of inheritance may be disturbed by unequal crossing over in male and female, by sex linkage, or by gametic selection. These cases are not discussed at all in the present paper; but it may be stated briefly that in the absence of selection the corresponding genetic algebras (of order n, say) possess train subalgebras (of order $n-1$).

The occurrence of a non-associative linear algebra in the simplest case of Mendelian inheritance was pointed out by Glivenko (1936).

§ 2. Notation.

By ***principal powers*** in a non-associative algebra, I mean powers in which the factors are absorbed one at a time always on the right or always on the left (see (3.6)). Otherwise, for the notation and nomenclature for non-associative products and powers, see my paper "On Non-Associative Combinations" (1939). The word ***pedigree*** which occurs there can now be interpreted almost in its ordinary biological sense.

Elements of a linear algebra (*i.e.* hypercomplex numbers) will be called ***elements*** and denoted by Latin letters, generally small ($a, b, \ldots$); but normalised elements, *i.e.* elements of unit weight (§ 3), will be denoted by Latin capitals (A, B, . . .). The letters m, n, r, however, denote positive integers.

Elements of the field **F** over which a linear algebra is defined will be called ***numbers*** and denoted by small Greek letters ($\alpha, \beta, \ldots$). Thus, an ***element*** is determined by its coefficients, which are ***numbers***. In the genetical applications, **F** may be taken as the field of real numbers. The enumerating indices (subscripts and superscripts) take positive integer values, either 1 to m, 1 to n, or 1 to r, according to the context.

Block capitals (**A**, **B**, . . .) denote algebras.

The symbol Σ indicates summation with respect to repeated indices, *e.g.* with respect to σ in (3.3), with respect to σ and τ in (5.3).

The symbol 1^μ stands for a set of 1's. Thus the formula (6.3) means the same as

$$\sum_{\sigma=1}^{n} \gamma_\sigma^{\mu\nu} = 1.$$

The advantage of this notation is that such formulæ retain their form under linear transformations of the basis of a genetic algebra, 1^μ being replaced by the vector ξ^μ (*cf.* (6.12)).

§ 3. Baric Algebras.

It is well known that a linear associative algebra possesses a matrix representation. Non-associative algebras in general do not, but may. The simplest such representation would be a scalar representation on the field **F** over which the algebra is defined. A linear algebra **X**, associative or not, which possesses a non-trivial representation of this kind, will be called *baric.*

The definition means that to any element x of **X** there corresponds a number $\xi(x)$ of **F**, not identically zero, such that

$$\xi(x+y) = \xi(x) + \xi(y), \quad \xi(\alpha x) = \alpha\xi(x), \quad \xi(xy) = \xi(x)\xi(y). \quad (x, y < \mathbf{X},\ \alpha < \mathbf{F}) \qquad (3.1)$$

$\xi(x)$ will be called the *weight* of x, or the *weight function* of **X**. If $\xi(x) \neq 0$, x can be *normalised*—that is, replaced by the element

$$\mathrm{X} = x/\xi(x) \quad . \quad . \quad . \quad . \quad . \quad (3.2)$$

of unit weight. Elements of zero weight will be called ***nil elements.*** The set **U** of all nil elements is evidently an invariant subalgebra of **X**; *i.e.* $\mathbf{XU} < \mathbf{U}$: it will be called the ***nil subalgebra.***

Let the multiplication table of a linear algebra **X** be

$$a^\mu a^\nu = \Sigma\gamma_\sigma^{\mu\nu}a^\sigma, \qquad (\mu, \nu, \sigma = 1, \ldots, n) \quad . \quad . \quad . \quad (3.3)$$

and let the general element be denoted

$$x = \Sigma\alpha_\mu a^\mu. \quad . \quad . \quad . \quad . \quad . \quad (3.4)$$

For **X** *to be a baric algebra, it is necessary and sufficient that the equations* (3.3), *regarded as ordinary simultaneous equations in* **F** *for the unknowns* a^μ, *should possess a non-null solution* $a^\mu = \xi^\mu$. For this is obviously necessary, the ξ^μ being the weights of the basic elements a^μ. Conversely, if the condition is satisfied and we take

$$\xi(x) = \Sigma\alpha_\mu\xi^\mu, \quad . \quad . \quad . \quad . \quad . \quad (3.5)$$

then (3.1) are at once deducible. The basic weights ξ^μ form the *weight vector* of **X**. In the genetical applications, $\xi^\mu = 1^\mu$.

Let the right rank equation (Dickson, 1914, § 19), or equation of lowest degree connecting the right principal powers,

$$x, \; x^2, \; x^3, \ldots, \qquad x^m = x^{m-1}x, \ldots, \quad . \quad . \quad . \quad (3.6)$$

be

$$f(x) \equiv x^r + \theta_1 x^{r-1} + \theta_2 x^{r-2} + \ldots + \theta_{r-1}x = 0, \quad . \quad . \quad (3.7)$$

where each coefficient θ_m is a homogeneous polynomial of degree m in the co-ordinates α_μ of x. Then $f(x)$, being zero, is of zero weight. Hence the equation is satisfied when we substitute $\xi(x)$ for x; consequently $x - \xi(x)$ must be a factor of $f(x)$. The same is true for the left rank equation. Thus

$$\xi(x) \textit{ is a root of the right and left rank equations.} \quad . \quad . \quad (3.8)$$

The weight function of an algebra is not necessarily unique. In fact, a commutative associative linear algebra for which the determinant $|\Sigma\gamma_\sigma^{\mu\nu}\gamma_\tau^{\sigma\tau}|$ does not vanish has n independent weight functions; and its rank equation is hence completely determined by (3.8) (Dickson, 1914, § 55, and the references given there).

§ 4. Train Algebras.

A baric algebra with the weight function $\xi(x)$ and right rank equation (3.7) will be called a ***right train algebra*** if the coefficients θ_m, in so far as they depend on the element x, depend only on $\xi(x)$. A *left train algebra* is defined similarly. For simplicity, suppose multiplication commutative, so that we may drop "left" and "right."

Since θ_m is homogeneous of degree m in the co-ordinates of x, it must in a train algebra be a numerical multiple of $\xi(x)^m$. Hence (if the field **F**

be sufficiently extended, *e.g.*, to include complex numbers) the rank equation can be factorised:

$$f(x) \equiv x(x-\xi)(x-\lambda_1\xi)(x-\lambda_2\xi)\ldots = 0. \qquad (4.1)$$

(It is implied that when the left side is expanded, powers of x are interpreted as principal powers.) The numbers $1, \lambda_1, \lambda_2 \ldots$ are the ***principal train roots*** of the algebra.

For a normalised element (3.7) becomes

$$f(X) \equiv X^r + \theta_1 X^{r-1} + \theta_2 X^{r-2} + \ldots + \theta_{r-1} X = 0, \qquad (4.2)$$

where now the θ's are constant (*i.e.* independent of X); and (4.1) becomes

$$f(X) \equiv X(X-1)(X-\lambda_1)(X-\lambda_2)\ldots = 0. \qquad (4.3)$$

Since (4.2) can be multiplied by X any number of times, it can be regarded as a linear recurrence equation with constant coefficients connecting the principal powers of the general normalised element X. Solving the recurrence relation for $X^m (m > r)$ in the usual way, we obtain $1, \lambda_1, \lambda_2 \ldots$ as the roots of the auxiliary equation; hence a formula for X^m can be written down in terms of $X, X^2, \ldots X^{r-1}$. Hence also for the general non-nil element $x = \xi X$, the value of $x^m = \xi^m X^m$ is known; while for a nil element u, $u^m = 0 (m > r)$.

The properties of train algebras will be studied elsewhere, and the following theorem proved:—

If (1) **X** *is a baric algebra*; (2) *its nil subalgebra* **U** *is nilpotent* (Wedderburn, 1908 *a*, p. 111); (3) *for* $m = 1, 2, 3, \ldots$, *the subalgebra* $\mathbf{U}^{(m)}$, *consisting of all products of altitude m* (Etherington, 1939, p. 156) *formed from nil elements is an invariant subalgebra of* **X** (*as it necessarily is of* **U**); *then* **X** *is a train algebra.*

For train algebras of rank $r = 2$ or 3, provided that the principal train roots do not include $\frac{1}{2}$, the conditions are necessary as well as sufficient; but I cannot say whether this converse holds more generally or not. I will call **X** a *special train algebra* if it satisfies the conditions (1), (2), (3). In such algebras it can be shown that there are many other sequences which have properties like those of the sequence of principal powers; *i.e.* sequences of elements derived from the general element, which satisfy linear recurrence equations whose coefficients, being functions of the weight only, become constants on normalisation. Such sequences will be called *trains*. For example, the sequence of plenary powers

$$x,\ x^2,\ x^{2\cdot 2},\ x^{2^3}, \ldots, \qquad (4.4)$$

and the sequence of primary products

$$x,\ Yx,\ Y.Yx,\ Y:Y.Yx, \ldots, \qquad (4.5)$$

form trains in a special train algebra.

It is convenient to denote the mth element of a train as $x^{[m]}$, and to regard it as a symbolic mth power of x. Let the normalised recurrence equation, or ***train equation***, be

$$g[X] \equiv X^{[s]} + \phi_1 X^{[s-1]} + \phi_2 X^{[s-2]} + \ldots + \phi_{s-1} X = 0, \qquad (4.6)$$

where the ϕ's are numerical constants. It is implied that the equation may be symbolically "multiplied all through" by X any number of times. It may also be symbolically factorised:

$$g[X] \equiv X[X-1][X-\mu_1][X-\mu_2] \ldots = 0. \qquad (4.7)$$

The square brackets indicate that after expansion powers of X are to be interpreted as symbolic powers. The expansion being performed as in ordinary algebra, multiplication of the symbolic factors is commutative and associative. Extra factors may be introduced without destroying the validity of the train equation; but assuming that all superfluous factors have been removed, s is the *rank* of the train, and the numbers $1, \mu_1, \mu_2, \ldots$ are the *train roots*, by means of which a formula for $X^{[m]}$ $(m > s)$ can be written down.

In the applications to genetics, it will be found that all the fundamental symmetrical genetic algebras are special train algebras. Various trains have genetical significance; the $X^{[m]}$ represent successive discrete generations of an evolving population or breeding experiment, and the train equation is the recurrence equation which connects them.

Thus, for example, plenary powers (4.4) refer to a population with random mating; principal powers (3.6) to a mating system in which each generation is mated back to one original ancestor or ancestral population; and the primary products (4.5) to the descendants of a single individual or subpopulation X mating at random within a population Y. Other mating systems are described by other sequences, and in various well-known cases these have the train property—that is, the determination of the mth generation depends ultimately on a linear recurrence equation with constant coefficients. It usually happens that the train roots are real, distinct, and not exceeding unity. Hence it may be shown that $X^{[m]}$ tends to equilibrium with increasing m; the rate of approach to equilibrium is ultimately that of a geometrical progression with common ratio equal to the largest train root excluding unity; but it may be some generations (depending on the number of train roots) before this rate of approach is manifest.

Train roots may be described as the eigen-values of the operation of symbolic multiplication by X, or in genetic language, the operation of passing from one generation to the next.

Train algebras of (principal) rank 3, which occur in several contexts

in genetics, have certain special properties. For example, *if the train equation for principal powers is* $X(X-1)(X-\lambda)=0$, *then the train equation for plenary powers is* $X[X-1][X-2\lambda]=0$; *and* vice versa. Examples may be seen below in (10.12), (12.4, 5), (15.3), where respectively $\lambda=0,\ \frac{1}{2}(1-\omega),\ \frac{1}{8}$.

§ 5. Duplication.

Let

$$a^\mu a^\nu = \Sigma\gamma_\sigma^{\mu\nu} a^\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.1)$$

be the multiplication table of a linear algebra $\mathbf{X}$ with basis a^μ $(\mu=1, \ldots, n)$. Then

$$a^\mu a^\nu \,.\, a^\theta a^\phi = \Sigma\gamma_\sigma^{\mu\nu} a^\sigma \,.\, \Sigma\gamma_\tau^{\theta\phi} a^\tau.$$

Writing

$$a^\mu a^\nu = a^{\mu\nu}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.2)$$

this becomes

$$a^{\mu\nu} a^{\theta\phi} = \Sigma\gamma_\sigma^{\mu\nu}\gamma_\tau^{\theta\phi} a^{\sigma\tau}, \quad . \quad . \quad . \quad . \quad . \quad (5.3)$$

which may be regarded as the multiplication table of another linear algebra, isomorphic with the totality of quadratic forms in the original algebra. It will be called the *duplicate* of $\mathbf{X}$, and denoted $\mathbf{X}'$. It is commutative and of order $\frac{1}{2}n(n+1)$ if $\mathbf{X}$ is commutative; non-commutative and of order n^2 if $\mathbf{X}$ is non-commutative. It is generally non-associative, even if $\mathbf{X}$ is associative. It is not to be confused with what may be called the *direct square* of $\mathbf{X}$, or direct product of two algebras isomorphic with $\mathbf{X}$: this would be an algebra of order n^2, having the multiplication table

$$a^{\mu\nu} a^{\theta\phi} = \Sigma\gamma_\sigma^{\mu\theta}\gamma_\tau^{\nu\phi} a^{\sigma\tau}, \quad . \quad . \quad . \quad . \quad . \quad (5.4)$$

differing from (5.3) in the arrangement of indices.

Some theorems on duplication will be proved elsewhere. It will be shown that the duplicates (i) of a linear transform of an algebra, (ii) of the direct product of two algebras, (iii) of a baric algebra with weight vector ξ^μ, (iv) of a train algebra with principal train roots $1, \lambda, \mu, \ldots$, are respectively (i) a linear transform of the duplicate algebra, (ii) the direct product of the duplicates, (iii) a baric algebra with weight vector $\xi^\mu\xi^\nu$, (iv) a train algebra with principal train roots $1, 0, \lambda, \mu, \ldots$ These theorems are relevant as follows: (iii) in view of §§ 7, 8; (ii) in view of § 9; (i) in connection with the method used in § 14; (iv) in deriving equations such as (10.10), (12.6).

Duplication of an algebra may be compared with the process of forming the second induced matrix of a given matrix (Aitken, 1935; *cf.* also Wedderburn, 1908 *b*).

§ 6. Gametic Algebras.

Consider the inheritance of characters depending on any number of gene differences at any number of loci on any number of chromosomes in a diploid or generally autopolyploid species. Assume that inheritance is symmetrical in the sexes: the sex chromosomes are thus excluded, and crossing over if present must be equal in male and female.

Let $G^1, G^2, \ldots, G^n$ denote the set of gametic types determined by these gene differences. Then there will be

$$m = \tfrac{1}{2}n(n+1) \qquad . \quad . \quad . \quad . \quad . \quad (6.1)$$

zygotic types $G^\mu G^\nu (= G^\nu G^\mu)$. The formulæ giving the series of gametic types produced by each type of individual, and their relative frequencies, may be written

$$G^\mu G^\nu = \Sigma \gamma_\sigma^{\mu\nu} G^\sigma, \qquad . \quad . \quad . \quad . \quad . \quad (6.2)$$

with the normalising conditions

$$\Sigma \gamma_\sigma^{\mu\nu} 1^\sigma = 1; \qquad . \quad . \quad . \quad . \quad . \quad (6.3)$$

$\gamma_\sigma^{\mu\nu}$ is then the probability that an arbitrary gamete produced by an individual of zygotic type $G^\mu G^\nu$ is of type G^σ.

(I speak of *zygotic types*—individuals distinguished by the gametes from which they were formed—rather than *genotypes*—individuals distinguished by the gametes which they produce—because the $G^\mu G^\nu$ are not all distinct genotypes if more than one chromosome is involved: the zygotic algebra, § 7, will have the same train equation if genotypes are used, but will then not be a duplicate algebra.)

A population P which produces gametes G^μ in proportions α_μ may be represented by writing

$$P = \Sigma \alpha_\mu G^\mu. \qquad . \quad . \quad . \quad . \quad . \quad (6.4)$$

Imposing the normalising condition

$$\Sigma \alpha_\mu 1^\mu = 1, \qquad . \quad . \quad . \quad . \quad . \quad (6.5)$$

α_μ denotes the probability that an arbitrary gamete produced by an arbitrary individual of P is of type G^μ.

A population may also be described by the proportions of the zygotic types $G^\mu G^\nu$ which it contains; thus we may write

$$P = \Sigma \alpha_{\mu\nu} G^\mu G^\nu, \qquad . \quad . \quad . \quad . \quad . \quad (6.6)$$

with the normalising condition

$$\Sigma \alpha_{\mu\nu} 1^\mu 1^\nu = 1, \qquad . \quad . \quad . \quad . \quad . \quad (6.7)$$

and a similar probability interpretation. We may suppose without loss of generality that $\alpha_{\mu\nu} = \alpha_{\nu\mu}$, so that in (6.6) the coefficient of $G^\mu G^\nu$ is

$2\alpha_{\mu\nu}$ if $\mu \neq \nu$. The two representations are connected by the gametic series formulæ (6.2); that is to say, from the zygotic representation (6.6) follows the gametic representation

$$P = \Sigma \alpha_{\mu\nu} \gamma_\sigma^{\mu\nu} G^\sigma. \qquad (6.8)$$

If two populations P, Q intermate at random, representations of the first filial generation are obtained by multiplying the gametic representations of P and Q; *i.e.* if

$$P = \Sigma \alpha_\mu G^\mu, \qquad Q = \Sigma \beta_\mu G^\mu, \qquad (6.9)$$

the population of offspring is

$$PQ = \Sigma \alpha_\mu \beta_\nu G^\mu G^\nu \qquad (6.10)$$

$$= \Sigma \alpha_\mu \beta_\nu \gamma_\sigma^{\mu\nu} G^\sigma. \qquad (6.11)$$

In particular, the population of offspring of random mating of P within itself is given by P^2.

We may now view the situation abstractly. The gametic series (6.2) form the multiplication table of a commutative non-associative linear algebra with basis $G^\mu (\mu = 1, \ldots, n)$. It will be called the *gametic algebra* for the type of inheritance considered, and denoted **G**. The equations (6.3) show that **G** is a baric algebra with weight vector

$$\xi^\mu = 1^\mu. \qquad (6.12)$$

With regard to its gametic type frequencies, a population is represented by a normalised element (6.4) of **G**. Multiplication in **G** has the significance described in § 1, and it follows from the multiplicative property of the weight in a baric algebra that PQ will be automatically normalised if P and Q are.

§ 7. Zygotic Algebras.

When individuals of types $G^\mu G^\nu$, $G^\theta G^\phi$ mate, the probability distribution of zygotic types in their offspring can be obtained by multiplying the gametic representations (given by (6.2)) together, and leaving the product in quadratic form (as in (6.10)). We obtain

$$G^\mu G^\nu \,.\, G^\theta G^\phi = \Sigma \gamma_\sigma^{\mu\nu} \gamma_\tau^{\theta\phi} G^\sigma G^\tau;$$

or, writing

$$Z^{\mu\nu} = G^\mu G^\nu \qquad (7.1)$$

to emphasise the union of paired gametes into single individuals,

$$Z^{\mu\nu} Z^{\theta\phi} = \Sigma \gamma_\sigma^{\mu\nu} \gamma_\tau^{\theta\phi} Z^{\sigma\tau}. \qquad (7.2)$$

These $\frac{1}{2}m(m+1)$ equations, then, are the formulæ giving the series of zygotic types produced by the mating type or *couple* $Z^{\mu\nu} \times Z^{\theta\phi}$, the

probability of $Z^{\sigma\tau}$ being the corresponding coefficient $\gamma_\sigma^{\mu\nu}\gamma_\tau^{\theta\phi}+\gamma_\tau^{\mu\nu}\gamma_\sigma^{\theta\phi}$ (if $\sigma\neq\tau$) or $\gamma_\sigma^{\mu\nu}\gamma_\tau^{\theta\phi}$ (if $\sigma=\tau$).

The linear algebra with basis $Z^{\mu\nu}$ and multiplication table (7.2) will be called the *zygotic algebra* for the type of inheritance considered. It is a baric algebra with weight vector $1^\mu 1^\nu$, the duplicate of the gametic algebra **G**, and will be denoted

$$\mathbf{Z}=\mathbf{G}'. \qquad (7.3)$$

A population, regarded as a distribution of zygotic types, is represented by a normalised element

$$P=\Sigma\alpha_{\mu\nu}Z^{\mu\nu}, \quad \text{where} \quad \Sigma\alpha_{\mu\nu}1^\mu 1^\nu=1;$$

and multiplication in **Z**, as in **G**, has the significance described in § 1. A product left in quadratic form in the Z's gives now the probability distribution of couples $Z^{\mu\nu}Z^{\theta\phi}$ among the parents; or, as I shall call it, the *copular representation* of the population of offspring.

§ 8. Further Duplicate Genetic Algebras.

The process of duplication can be applied repeatedly. Thus the $\frac{1}{2}m(m+1)$ types of paired zygotes, or couples,

$$K^{\mu\nu.\theta\phi}=Z^{\mu\nu}Z^{\theta\phi}, \qquad (8.1)$$

can be taken as the basis of a new linear algebra

$$\mathbf{K}=\mathbf{Z}'=\mathbf{G}''. \qquad (8.2)$$

Call it the *copular algebra*. A normalised element with positive coefficients

$$P=\Sigma\alpha_{\mu\nu.\theta\phi}K^{\mu\nu.\theta\phi}, \quad \text{where} \quad \Sigma\alpha_{\mu\nu.\theta\phi}1^\mu 1^\nu 1^\theta 1^\phi=1,$$

is the copular representation of a population—the probability distribution of couples in the parents of the individuals comprised in the population.

Similarly, in the next duplicate algebra **K**', the basic symbols would classify tetrads of grandparents.

In all these algebras, multiplication has the significance described in § 1.

§ 9. Combination of Genetic Algebras.

Consider two distinct genetic classifications referring to the same population P, firstly into a set of m genetic types

$$A^1, A^2, \ldots, A^m;$$

secondly into a set of n genetic types

$$B^1, B^2, \ldots, B^n$$

of the same kind (gametic, zygotic, etc.). Let the corresponding genetic algebras be **A**, **B** with multiplication tables

$$A^\mu A^\nu = \Sigma \gamma_\sigma^{\mu\nu} A^\sigma, \qquad B^\theta B^\phi = \Sigma \delta_\tau^{\theta\phi} B^\tau.$$

By taking account of both classifications at once, we obtain a third classification which may be called their *product*, into mn genetic types

$$C^{\mu\theta} = A^\mu B^\theta.$$

The type $C^{\mu\theta}$ comprises all individuals (gametes, zygotes, etc.) who are of type A^μ in the first classification, B^θ in the second.

If the characters of the two classifications are inherited independently, *i.e.* if they involve two quite distinct sets of chromosomes, then the probabilities $\gamma_\sigma^{\mu\nu}$, $\delta_\tau^{\theta\phi}$ refer to independent events. Hence the genetic algebra with basis $C^{\mu\theta}$ is the direct product

$$\mathbf{C} = \mathbf{AB};$$

i.e. its multiplication table is

$$C^{\mu\theta} C^{\nu\phi} = \Sigma \gamma_\sigma^{\mu\nu} \delta_\tau^{\theta\phi} C^{\sigma\tau}.$$

It follows that a genetic algebra which depends on several autosomal linkage groups must be a direct product **ABC** . . . of genetic algebras, one factor algebra for each linkage group.

If, however, the A and B classifications are independent but genetically linked, *i.e.* if they involve two quite distinct sets of gene loci but not distinct sets of chromosomes, then the probabilities $\gamma_\sigma^{\mu\nu}$, $\delta_\tau^{\theta\phi}$ are not independent. Regarded as a linear set, **C** is still the product of the linear sets **A** and **B**; but the algebra **C** will not be the direct product of the algebras **A** and **B** (except in the very exceptional case when all crossing over values between A and B are precisely 50 per cent.). It is, however, still the case that **C** contains subalgebras isomorphic with **A** and **B**. For example, if these algebras are gametic, and if we keep the first index of $C^{\mu\theta}$ constant, we are virtually disregarding all the A-loci, so we obtain a subalgebra isomorphic with **B**; and this can be done in m ways.

Hence a genetic algebra based on the allelomorphs of several autosomal loci possesses numerous automorphisms.

It will be shown in § 14 that even when linkage is involved the gametic algebra can be symbolically factorised, and regarded as a symbolic direct product of non-commutative factor algebras, one for each locus (see (14.12)).

§§ 10–15. Examples of Symmetrical Genetic Algebras.

A more detailed description of practical applications will be given elsewhere. My object here is simply to show that the genetic algebras are

train algebras. I give in each case the principal and plenary train equations, *i.e.* the identities of lowest degree connecting respectively the sequences of principal and plenary powers of a normalised element. As explained in § 4, these are really recurrence equations, and have a special significance in genetics.

§ 10. Simple Mendelian Inheritance.

For a single autosomal gene difference (D, R), the gametic multiplication table is

$$DD = D, \qquad DR = \tfrac{1}{2}D + \tfrac{1}{2}R, \qquad RR = R. \qquad (10.1)$$

Writing

$$A = DD, \qquad B = DR, \qquad C = RR, \qquad (10.2)$$

we find, *e.g.*,

$$B^2 = (\tfrac{1}{2}D + \tfrac{1}{2}R)^2 = \tfrac{1}{4}A + \tfrac{1}{2}B + \tfrac{1}{4}C.$$

Hence and similarly the zygotic multiplication table is

$$\begin{gathered} A^2 = A, \qquad B^2 = \tfrac{1}{4}A + \tfrac{1}{2}B + \tfrac{1}{4}C, \qquad C^2 = C, \\ BC = \tfrac{1}{2}B + \tfrac{1}{2}C, \qquad CA = B, \qquad AB = \tfrac{1}{2}A + \tfrac{1}{2}B. \end{gathered} \qquad (10.3)$$

Call these two algebras $\mathbf{G}_2$, $\mathbf{Z}_2$ ($\mathbf{Z}_2 = \mathbf{G}_2'$), and denote their general elements

$\mathbf{G}_2$: $$x = \delta D + \rho R, \qquad (10.4)$$

$\mathbf{Z}_2$: $$x = \alpha A + 2\beta B + \gamma C. \qquad (10.5)$$

The principal rank equations are

$\mathbf{G}_2$: $$x^2 - (\delta + \rho)x = 0, \qquad (10.6)$$

$\mathbf{Z}_2$: $$x^3 - (\alpha + 2\beta + \gamma)x^2 = 0; \qquad (10.7)$$

and the plenary rank equations (or identities of lowest degree connecting plenary powers of the general elements) are (10.6) and

$\mathbf{Z}_2$: $$x^{2\cdot 2} - (\alpha + 2\beta + \gamma)^2 x^2 = 0. \qquad (10.8)$$

A population P is represented by an element of unit weight in either algebra, *i.e.* (10.4) or (10.5) with

$$\delta + \rho = 1, \qquad \alpha + 2\beta + \gamma = 1,$$

the ratios $\delta : \rho$, $\alpha : 2\beta : \gamma$ giving the relative frequencies of the gametic types which it produces or genotypes which it contains. In this case (10.6), (10.7), (10.8) become the train equations

$\mathbf{G}_2$: $$P^2 = P, \qquad (10.9)$$

$\mathbf{Z}_2$: $$P^3 = P^2, \qquad P^{2\cdot 2} = P^2, \qquad (10.10)$$

expressing facts well known in genetics. It is convenient to write these equations in the form (*cf.* 4.7)

$$\mathbf{G}_2: \qquad P(P-1)=0, \qquad (10.11)$$

$$\mathbf{Z}_2: \qquad P^2(P-1)=0, \qquad P^{2}[P-1]=0. \qquad (10.12)$$

§ 11. Multiple Allelomorphs.

For n allelomorphs $G^\mu(\mu=1, \ldots, n)$, the gametic and zygotic multiplication tables are

$$G^\mu G^\nu = \tfrac{1}{2}G^\mu + \tfrac{1}{2}G^\nu, \qquad (11.1)$$

$$Z^{\mu\nu}Z^{\theta\phi} = \tfrac{1}{4}Z^{\mu\theta} + \tfrac{1}{4}Z^{\mu\phi} + \tfrac{1}{4}Z^{\nu\theta} + \tfrac{1}{4}Z^{\nu\phi}, \qquad (11.2)$$

where $Z^{\mu\nu}=G^\mu G^\nu$. The algebras $\mathbf{G}_n$, $\mathbf{Z}_n$ so determined reduce to $\mathbf{G}_2$, $\mathbf{Z}_2$ when $n=2$; and they have in general the same train equations (10.11), (10.12).

§ 12. Linked Allelomorphs.

For two linked series of multiple allelomorphs, respectively m and n in number, with crossing over probability ω, the gametic multiplication table is

$$G^{\mu\alpha}G^{\nu\beta} = \tfrac{1}{2}(1-\omega)(G^{\mu\alpha}+G^{\nu\beta}) + \tfrac{1}{2}\omega(G^{\mu\beta}+G^{\nu\alpha}), \qquad (12.1)$$

where $G^{\mu\alpha}(\mu=1, \ldots, m;\ \alpha=1, \ldots, n)$ are the mn gametic types. Denote this gametic algebra $\mathbf{G}_{mn}(\omega)$. The principal and plenary rank equations are

$$x^3 - \tfrac{1}{2}(3-\omega)\xi x^2 + \tfrac{1}{2}(1-\omega)\xi^2 x = 0, \qquad (12.2)$$

$$x^{2\cdot 2} - (2-\omega)\xi^2 x^2 + (1-\omega)\xi^3 x = 0, \qquad (12.3)$$

giving for a normalised element P the train equations

$$P^3 - \tfrac{1}{2}(3-\omega)P^2 + \tfrac{1}{2}(1-\omega)P \equiv P(P-1)\left(P - \frac{1-\omega}{2}\right) = 0, \qquad (12.4)$$

$$P^{2\cdot 2} - (2-\omega)P^2 + (1-\omega)P \equiv P[P-1][P-(1-\omega)] = 0. \qquad (12.5)$$

In the duplicate algebra $\mathbf{Z}_{mn}(\omega) = \mathbf{G}_{mn}(\omega)'$ the corresponding equations are

$$P^2(P-1)\left(P - \frac{1-\omega}{2}\right) = 0, \qquad P^{2}[P-1][P-(1-\omega)] = 0. \qquad (12.6)$$

§ 13. Independent Allelomorphs.

Consider two series of multiple allelomorphs in separate autosomal linkage groups. This being indistinguishable from the case of § 12 with $\omega=\tfrac{1}{2}$, the gametic algebra is $\mathbf{G}_{mn}(\tfrac{1}{2})$. As in § 9, it may also be expressed as the direct product $\mathbf{G}_m\mathbf{G}_n$.

§ 14. Linkage Group.

I will first rewrite equations (12.1) with a change of notation. I will then write down the analogous equations for the case of three linked loci, and examine the structure of the corresponding algebra. This will be a sufficient indication of the procedure which can be followed out quite generally for a complete linkage group comprising any number of loci on one autosome, with any number of allelomorphs at each locus. The method may be extended to include any number of linkage groups.

Equations (12.1) may be written

$$\mathrm{AB}\,.\,\mathrm{A'B'}=\tfrac{1}{2}(1-\omega)(\mathrm{AB}+\mathrm{A'B'})+\tfrac{1}{2}\omega(\mathrm{AB'}+\mathrm{A'B}). \quad . \quad . \quad (14.1)$$

Here A and B refer to the two gene loci. $\mathrm{A}^{\mu}\mathrm{B}^{\alpha}$ would mean the same as $\mathrm{G}^{\mu\alpha}$—a gamete with the μth allelomorph at the A-locus and the αth at B; but dropping the indices AB and A′B′ stand for any particular gametic types, the same or different.

(14.1) may again be rewritten

$$\mathrm{AB}\,.\,\mathrm{A'B'}=\tfrac{1}{2}\varpi(\mathrm{A}+\chi\mathrm{A'})(\mathrm{B}+\chi\mathrm{B'}), \quad . \quad . \quad . \quad (14.2)$$

where $\varpi=1-\omega$ and χ is an operator which interchanges ω and ϖ, so that $\chi^2=1$ and $\varpi\chi=\omega$.

Now consider the case of three loci A, B, C, having respectively m, n, r allelomorphs, and crossing over probabilities ω_{AB}, ω_{BC}, ω_{AC}. The gametic algebra may be symbolised conveniently as $\mathbf{G}_{mnr}(\omega)$, where ω is the symmetrical matrix of the crossing over values, with diagonal zeros. Its multiplication table, comprising $\tfrac{1}{2}mnr(mnr+1)$ formulæ, is

$$\begin{aligned}\mathrm{ABC}\,.\,\mathrm{A'B'C'}=\tfrac{1}{2}\lambda(\mathrm{ABC}+\mathrm{A'B'C'})+\tfrac{1}{2}\mu(\mathrm{A'BC}+\mathrm{AB'C'})\\ +\tfrac{1}{2}\nu(\mathrm{AB'C}+\mathrm{A'BC'})+\tfrac{1}{2}\rho(\mathrm{ABC'}+\mathrm{A'B'C}),\end{aligned} \quad (14.3)$$

where

$$\lambda+\mu+\nu+\rho=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.4)$$

$$\mu+\nu=\omega_{\mathrm{AB}}, \qquad \nu+\rho=\omega_{\mathrm{BC}}, \qquad \mu+\rho=\omega_{\mathrm{AC}}. \quad . \quad . \quad (14.5)$$

The ω's are not independent, but are connected only by an inequality (Haldane, 1918):

$$\omega_{\mathrm{AC}}=\omega_{\mathrm{AB}}+\omega_{\mathrm{BC}}-\kappa\omega_{\mathrm{AB}}\omega_{\mathrm{BC}}, \quad \text{where} \quad 0\leqslant\kappa\leqslant 2, \quad . \quad . \quad (14.6)$$

from which may be deduced

$$\mu\rho\geqslant\nu\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.7)$$

Now introduce the following operators:—

$$\left.\begin{array}{llll}\chi_1 & \text{interchanges } \lambda \text{ with } \mu, & \nu \text{ with } \rho, \\ \chi_2 & \quad ,, \quad \lambda \;,,\; \nu, & \rho \;,,\; \mu, \\ \chi_3 & \quad ,, \quad \lambda \;,,\; \rho, & \mu \;,,\; \nu.\end{array}\right\} \quad . \quad . \quad (14.8)$$

Together with 1, they form an Abelian group, having the relations

$$\left.\begin{array}{c}\chi_2\chi_3=\chi_1, \quad \chi_3\chi_1=\chi_2, \quad \chi_1\chi_2=\chi_3, \\ \chi_1^2=\chi_2^2=\chi_3^2=\chi_1\chi_2\chi_3=1.\end{array}\right\} \qquad (14.9)$$

(14.3) may then be rewritten:

$$\mathrm{ABC}\,.\,\mathrm{A'B'C'}=\tfrac{1}{2}\lambda(\mathrm{A}+\chi_1\mathrm{A'})(\mathrm{B}+\chi_2\mathrm{B'})(\mathrm{C}+\chi_3\mathrm{C'}). \qquad (14.10)$$

This symbolism can be manipulated with considerable freedom. For example, an expression such as $(\alpha\mathrm{ABC}+\beta\mathrm{A'BC})$ can be written $(\alpha\mathrm{A}+\beta\mathrm{A'})\mathrm{BC}$; and when two such expressions are multiplied, the distributive law works. The interchange symbols co-operate in the same way.

(14.10) may again be rewritten

$$\mathrm{ABC}\,.\,\mathrm{A'B'C'}=(\chi_0\mathrm{A}+\chi_1\mathrm{A'})(\chi_0\mathrm{B}+\chi_2\mathrm{B'})(\chi_0\mathrm{C}+\chi_3\mathrm{C'}), \qquad (14.11)$$

where $\chi_0=1$, and the operand $\frac{1}{2}\lambda$ is implied. Finally, (14.11) may be analysed into

$$\mathrm{AA'}=\chi_0\mathrm{A}+\chi_1\mathrm{A'}, \qquad \mathrm{BB'}=\chi_0\mathrm{B}+\chi_2\mathrm{B'}, \qquad \mathrm{CC'}=\chi_0\mathrm{C}+\chi_3\mathrm{C'}. \quad (14.12)$$

This separation of the symbols, or factorisation of the algebra (*cf.* end of § 9), will evidently yield valid results, provided that after recombination and application of (14.9), χ_0 is interpreted as $\frac{1}{2}\lambda$, χ_1 as $\frac{1}{2}\mu$, χ_2 as $\frac{1}{2}\nu$, χ_3 as $\frac{1}{2}\rho$. It must be noted that the symbols when separated in this way are non-commutative; *e.g.* $\mathrm{AA'}\neq\mathrm{A'A}$, since $\mathrm{ABC}\,.\,\mathrm{A'B'C'}\neq\mathrm{A'BC}\,.\,\mathrm{AB'C'}$.

Select a particular gametic type **ABC**, and write

$$\mathbf{A}-\mathrm{A}=u, \qquad \mathbf{B}-\mathrm{B}=v, \qquad \mathbf{C}-\mathrm{C}=w, \qquad (14.13)$$

where $\mathrm{A}\neq\mathbf{A}$, $\mathrm{B}\neq\mathbf{B}$, $\mathrm{C}\neq\mathbf{C}$. Thus the symbols u, v, w are nil elements having respectively $m-1$, $n-1$, $r-1$ possible values. We have from (14.12):

$$\mathbf{A}^2=(\chi_0+\chi_1)\mathbf{A},$$
$$\mathbf{A}u=\mathbf{A}^2-\mathbf{A}\mathrm{A}=(\chi_0+\chi_1)\mathbf{A}-(\chi_0\mathbf{A}+\chi_1\mathrm{A})=\chi_1 u,$$
$$u\mathbf{A}=\mathbf{A}^2-\mathrm{A}\mathbf{A}=(\chi_0+\chi_1)\mathbf{A}-(\chi_0\mathrm{A}+\chi_1\mathbf{A})=\chi_0 u,$$
$$u^2=\mathbf{A}^2-\mathbf{A}\mathrm{A}-\mathrm{A}\mathbf{A}+\mathrm{A}^2=(\chi_0+\chi_1)\mathbf{A}-(\chi_0\mathbf{A}+\chi_1\mathrm{A})-(\chi_0\mathrm{A}+\chi_1\mathbf{A})+(\chi_0+\chi_1)\mathrm{A}=0,$$

and eight similar equations.

Now write

$$\left.\begin{array}{llll}\mathbf{ABC}=\mathrm{I}, & u\mathbf{BC}=\bar{u}, & \mathbf{A}v\mathbf{C}=\bar{v}, & \mathbf{AB}w=\bar{w}, \\ \mathbf{A}vw=\overline{vw}, & u\mathbf{B}w=\overline{wu}, & uv\mathbf{C}=\overline{uv}, & uvw=\overline{uvw}.\end{array}\right\} \qquad (14.14)$$

The symbols I, $\bar{u}$, $\bar{v}$, $\bar{w}$, $\overline{vw}$, $\overline{wu}$, $\overline{uv}$, $\overline{uvw}$ thus introduced are linear and linearly independent in the gametic type symbols; and their number is

$$1+(m-1)+(n-1)+(r-1)+(n-1)(r-1)+(r-1)(m-1)+(m-1)(n-1)$$
$$+(m-1)(n-1)(r-1)=mnr,$$

which is equal to the number of gametic type symbols. They may thus be taken as a new basis for the gametic algebra. The transformed multiplication table is then easily deduced. We find, for example,

$$\mathrm{I}^2 = \mathrm{I},$$
$$\mathrm{I}\bar{u} = \mathbf{A}u \cdot \mathbf{B}^2 \cdot \mathbf{C}^2 = \chi_1(\chi_0+\chi_2)(\chi_0+\chi_3)\bar{u} = (\chi_0+\chi_1+\chi_2+\chi_3)\bar{u} = \tfrac{1}{2}\bar{u},$$

since $\chi_0+\chi_1+\chi_2+\chi_3$ is to be interpreted as $\frac{1}{2}\lambda+\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}\rho=\frac{1}{2}$. Similarly:

$$\mathrm{I}\overline{vw} = \tfrac{1}{2}(\lambda+\mu)\overline{vw}, \qquad \mathrm{I}\overline{uvw} = \tfrac{1}{2}\lambda\overline{uvw},$$
$$\bar{u}\bar{v} = \tfrac{1}{2}(\nu+\mu)\overline{uv}, \qquad \bar{u}\overline{vw} = \tfrac{1}{2}\mu\overline{uvw}, \qquad \bar{u}^2 = \bar{u}\overline{uv} = \bar{u}\overline{uvw} = 0.$$

These results are typical, all other products in the transformed multiplication table being obtainable from them by cyclic permutation of u, v, w and μ, ν, ρ and 1, 2, 3.

It is now readily verifiable that the algebra has the structure of a special train algebra as defined in § 4, with

$$\mathsf{U} = (\bar{u}, \bar{v}, \bar{w}, \overline{vw}, \overline{wu}, \overline{uv}, \overline{uvw}), \qquad \mathsf{U}^{(1)} = (\overline{vw}, \overline{wu}, \overline{uv}, \overline{uvw}),$$
$$\mathsf{U}^{(2)} = (\overline{uvw}), \qquad \mathsf{U}^{(3)} = 0.$$

Many of its properties can be most easily deduced from this transformed form. It can be shown that its principal and plenary train roots, other than unity, are the results of

$$\chi_0, \qquad \chi_0+\chi_1, \qquad \chi_0+\chi_2, \qquad \chi_0+\chi_3,$$

operating respectively on $\frac{1}{2}\lambda$ and λ. Further details are postponed until the properties of special train algebras have been studied elsewhere.

§ 15. Polyploidy.

A single example—the simplest possible—will illustrate the occurrence of special train algebras in this connection. The gametic algebra with multiplication table

$$\left.\begin{array}{lll} \mathrm{A}^2=\mathrm{A}, & \mathrm{B}^2=\mathrm{AC}=\frac{1}{6}\mathrm{A}+\frac{2}{3}\mathrm{B}+\frac{1}{6}\mathrm{C}, & \\ \mathrm{C}^2=\mathrm{C}, & \mathrm{BC}=\frac{1}{2}\mathrm{B}+\frac{1}{2}\mathrm{C}, & \mathrm{AB}=\frac{1}{2}\mathrm{A}+\frac{1}{2}\mathrm{B}, \end{array}\right\} \qquad (15.1)$$

refers to the inheritance of a single autosomal gene difference in autotetraploids. (*Cf.* Haldane, 1930, the case $m=2$, with A, B, C written for A^2, $\mathrm{A}a$, a^2.)

This is a special train algebra, as may be seen by performing the transformation

$$\mathrm{A}=\mathrm{A}, \qquad \mathrm{A}-\mathrm{B}=u, \qquad \mathrm{A}-2\mathrm{B}+\mathrm{C}=p. \qquad (15.2)$$

It has the principal and plenary train equations

$$\mathrm{P}(\mathrm{P}-1)(\mathrm{P}-\tfrac{1}{3})=0, \qquad \mathrm{P}[\mathrm{P}-1][\mathrm{P}-\tfrac{1}{3}]=0. \qquad (15.3)$$

Summary.

A population can be classified genetically at various levels, according to the frequencies of the gametic types which it produces, of the zygotic types of individuals which it contains, of types of mating pairs in the preceding generation, and so on. It is represented accordingly by means of hypercomplex numbers in one or other of a series of linear algebras (gametic, zygotic, copular, . . .), each algebra being isomorphic with the quadratic forms of the preceding algebra. Such a series of *genetic algebras* exists for any mode of genetic inheritance which is symmetrical in the sexes. (Genetic algebras for unsymmetrical inheritance also exist, but are not considered here.) Many calculations which occur in theoretical genetics can be expressed as manipulations within these algebras.

The algebras which arise in this way are all commutative non-associative linear algebras of a special kind. Firstly, they are *baric algebras*, *i.e.* they possess a scalar representation; secondly, they are *train algebras*, *i.e.* the rank equation of a suitably normalised hypercomplex number has constant coefficients. Some theorems concerning such algebras are enunciated.

REFERENCES TO LITERATURE.

Aitken, A. C., 1935. "The normal form of compound and induced matrices," *Proc. Lond. Math. Soc.* (2), vol. xxxviii, pp. 354–376.

Dickson, L. E., 1914 (reprinted 1930). *Linear algebras*, Cambridge Tract, No. 16.

Etherington, I. M. H., 1939. "On non-associative combinations," *Proc. Roy. Soc. Edin.*, vol. lix, pp. 153–162.

Geppert, H., and Koller, S., 1938. *Erbmathematik*, Leipzig.

Glivenko, V., 1936. "Algèbre mendelienne," *Moscow Acad. Sci. C.R.*, vol. iv, pp. 385–386.

Haldane, J. B. S., 1918. "The combination of linkage values," *Journ. Gen.*, vol. viii, pp. 299–309.

——, 1930. "Theoretical genetics of autopolyploids," *Journ. Gen.*, vol. xxii, pp. 359–372.

Jennings, H. S., 1935. *Genetics*, London.

Wedderburn, J. H. M., 1908 (*a*). "On hypercomplex numbers," *Proc. Lond. Math. Soc.* (2), vol. vi, pp. 77–118.

——, 1908 (*b*). "On certain theorems in determinants," *Proc. Edin. Math. Soc.* (1), vol. xxvii, pp. 67–69.

(*Issued separately November* 9, 1939.)

OBITUARY NOTICES.

Henry Edward Armstrong, Ph.D., LL.D., F.R.S., Hon. F.R.S.E.

HENRY EDWARD ARMSTRONG, who was elected an Honorary Fellow in 1934, died on July 13, 1937, in his ninetieth year. Although his connection with our Society was brief, he had been a Fellow of the Chemical Society since 1870 and of the Royal Society of London since 1876. For many years, indeed, he had been recognised as the "grand old man" of British Chemistry.

He first studied chemistry under Hoffmann at the Royal College of Chemistry in 1865; Tyndall and Huxley were also his scientific instructors. In 1868 he left the private laboratory of Frankland to obtain his Ph.D. degree with Kolbe at Leipzig. He inherited there Kolbe's gift of provocative criticism, for the skilful employment of which he will always be remembered.

There followed a long teaching and research career at the London Institution, Finsbury Square, and at the City and Guilds College, South Kensington. As a teacher, Armstrong was characteristically unorthodox, and he disturbed his complacent colleagues for decades by his constant advocacy of what became known as the "neuristic method" of presenting science experimentally in schools, as opposed to the traditional "didactic method." In research Armstrong was pre-eminent in organic chemistry, and his inspiration is evident by the large number of research students who worked under his direction and later became leaders in chemical industry or education.

As a controversialist, Armstrong knew no equal. For fifty years he never ceased to attack the Arrhenius theory of ionization in solution with almost religious fervour. He had himself carried out a most extensive study of the physical properties of sulphuric acid just before the ionic hypothesis came into prominence, and his communications frequently read as if he were still dipping his pen into that liquid. Never, however, was there any personal rancour in his polemics; he could be just as genial in conversation as vituperative in writing.

There can be no doubt that as *laudator temporis acti* he frequently failed to appreciate the significance of new lines of chemical advance, but there can also be no doubt that he frequently acted as a most efficient and salutary brake on over-fanciful speculations.

J. K.

George Barger, M.A., D.Sc., Hon. D.Sc., Hon. M.D., LL.D., F.R.S.

PROFESSOR BARGER came to Edinburgh in 1919 and was elected a Fellow of the Royal Society of Edinburgh in 1922. He served on the Council from 1925 to 1928.

When he joined the University of Edinburgh he already was recognised as one of the leading organic chemists in the country and his professional reputation increased steadily during his occupancy of the Chair of Chemistry in relation to Medicine which he held for nearly twenty years. He attracted around him a group of active research workers, and his school became one of the most important centres in the country of research in organic chemistry.

The best known piece of work produced in this school was the establishment of the correct structural formula of thyroxine and its subsequent synthesis. Another important synthesis, that of vitamin B, was also achieved, but in this case an alternative method discovered elsewhere had priority in publication.

The Department was particularly concerned with the structure of alkaloids and produced a large mass of important work on this subject. A separate and important line of work was that of Stedman on the structure of physostigmine and on choline esterase. These examples suffice to indicate the wide range of activity of the school which Professor Barger founded and inspired.

He received many degrees from foreign universities and was member of a number of foreign academies. He gave the Dohme Lectures in Baltimore in 1928, and these were published in the form of a masterly monograph on "Ergot and Ergotism." Two awards of particular distinction were the Longstaff Medal of the Chemical Society in 1936 and the Davy Medal of the Royal Society in 1938.

Any account of his work would be incomplete without reference to his wide international interests. These were favoured by exceptional linguistic abilities; indeed it was said that he could deliver a lecture in most of the important European languages. He performed a notable service to the cause of international science on the occasion of the holding in Edinburgh of the International Congress of Physiology in 1923. Professor Barger was secretary and Sir Edward Sharpey Schafer was

president, and together they succeeded, against a certain opposition, in making the Congress truly international by issuing invitations to the Germans and Austrians. This was the first important scientific congress after the war at which full international relations were resumed, and it is pleasant to remember that our city had this honour. Professor Barger travelled widely and was in close touch with all the important Continental laboratories. One always felt that he represented in an outstanding manner the old tradition of the international scholar who was at home everywhere where learning was held in esteem. His premature death came as the greater shock to his friends because his activity in mind and body always made him appear younger than his years.

In 1937 he left Edinburgh to take the Chair of Chemistry at the University of Glasgow. He was nearly sixty, and his willingness to make a change of such importance at this age indicates his unusual mental energy. During his short tenure of office at Glasgow he was kept very busy by the work consequent on the reconstruction of the Department.

The Royal Society of Edinburgh has lost one of its most distinguished figures, for in Britain Professor Barger was recognised as one of the most distinguished organic chemists in the country and abroad he was one of the best known of our men of science. His intellectual gifts were allied with an outstanding character. The qualities which his colleagues associate with his memory are high intellectual distinction, an exceptionally clear and critical intelligence and a unique simplicity and directness of thought and purpose. These qualities not only endeared him to his colleagues but also made him one whose opinion always commanded respect.

He died on January 5, 1939.

See also *Obituary Notices of the Fellows of The Royal Society*, vol. iii, No. 8, 1940, pp. 63–85.

A. J. C.

Sir James Barr, C.B.E., M.D., LL.D., F.R.C.P.

JAMES BARR, who died on November 16, 1938, at the age of eighty-nine, was born in County Tyrone, Ulster. He was educated at Londonderry and the University of Glasgow, where he graduated in 1873. Appointed house physician to the David Lewis Northern Hospital, Liverpool, he later succeeded Sir William Mitchell Banks as medical officer to the Kirkdale Gaol. The reforms which he suggested in the care and treatment of prisoners were widely recognised and at the request of the Government he visited and reported upon the conditions obtaining in the Irish prisons. His work in this connection was acknowledged by a Knighthood which he received in 1905.

Sir James was associated prominently with the medical and social life of Liverpool for many years. He was an Honorary Physician on the staff of the Royal Infirmary and was President of the Liverpool Medical Institute. During the Great War he acted as chairman of the Liverpool War Committee and was awarded a C.B.E. for his services. A Knight of Grace of the Order of St John of Jerusalem; he was an LL.D. of Liverpool and Toronto.

He was elected a Fellow in 1904.

E. B.

Harvey (Williams) Cushing, For. Mem. R.S., Hon. F.R.S.E.

HUMANITY in general and medical science in particular have cause to regret the passing of one of the truly great men of our time—Dr Harvey Cushing.

He came of an illustrious medical family. Receiving his medical education at Boston, he migrated to Baltimore, where he came under the influence of the remarkable group of great medical personalities who then led the Johns Hopkins School—Halstead and Osler in particular. At this period he made his first visits to European centres, which were frequently repeated in later years. From 1912 to 1932 he occupied the Moseley Chair of Surgery at Harvard. His latter years were spent at Yale University.

After acquiring post-graduate clinical experience, he turned to the laboratory of experimental physiology. It is clear that from an early period he focussed his mind on the nervous system and with it the minute, obscure, and then unnoticed pituitary gland. With the imaginative insight characteristic of the great he concentrated on this latter problem, and when death finally intervened he had taken a major part in elucidating this controller of primitive and fundamental bodily activities in all its scientific aspects and connections, and had brought recognition and surgical treatment of its disorders to a high standard of efficiency. His intensive studies and progressive perfection of surgical treatment of one nervous disease group or entity after another is an awe-inspiring model of a life's work ordered by a master mind. The native genius, the courage, the iron will, the ability to recognise and seize opportunity, which combined for this achievement may be inferred from its magnitude.

Few have commanded such respect, admiration, and hero-worship from their pupils and associates as did Cushing; his influence on medical science and surgery to-day through his personal disciples is even more potent and nearly as widespread as through his extensive writings. Though with his mind ever bent on major objectives, his patients were his adoring friends; no detail concerning their welfare escaped his notice.

In spite of his great scientific activities, interests and ambitions, he felt the call of the Great War, and served with the American Ambulance

in France in 1915 and again from 1917–1919. While his activities were largely medical his *Life of Sir William Osler* is a notable contribution to general literature. His chief interest outside scientific medicine was in the history of medicine—he delighted to commune with the great medical personalities of past centuries.

His merit was recognised by learned institutions and societies in every country and their honours were showered upon him.

In his last years, though crippled from time to time by an affection of the legs, his output of contributions to medical science rose in quality and volume almost to the end.

His final illness lasted but a few days. His end—almost as if controlled in time and mode by the great mind that planned his life—leaves those who knew him to mourn a dear friend, but far above this sorrow is the inspiration of his vital personality and great achievement.

He was elected an Honorary Fellow in 1939, and died on October 7, 1939.

N. M. D.

Sir Frank Watson Dyson, K.B.E., D.Sc., LL.D., F.R.S.

SIR FRANK WATSON DYSON was born on January 8, 1868, at Ashby, Lincolnshire. He was educated at Bradford Grammar School and proceeded to Trinity College, Cambridge, graduating as second Wrangler in the Mathematical Tripos of 1889. In 1891 he was awarded a Smith's Prize and was elected a Fellow of Trinity, and in 1892 he was elected to an Isaac Newton Studentship. It is interesting to note that the first two holders of this studentship subsequently came to Edinburgh as Astronomers Royal for Scotland, the first Isaac Newton Student being R. A. Sampson who was elected in 1891.

At this period Dyson was engaged in research on potential theory, and one of his earlier papers was on "The Motion of a Satellite round a Spheroidal Planet." But in 1894 he was appointed Chief Assistant at the Royal Observatory, Greenwich, a position which he held until 1905. During this period he was intimately concerned with work on the Greenwich section of the Astrographic Catalogue, but his most important contribution to astronomical knowledge was the reduction, in collaboration with W. G. Thackeray, of the transit observations of more than 4000 circumpolar stars, made by Stephen Groombridge at Blackheath between the years 1806 and 1816. By comparison with modern observations these observations yielded proper motions of high accuracy, and thus provided material for the investigation of problems relating to stellar motions. The results of this heavy piece of work were published in 1905 and their appearance was most opportune, for in the previous year Kapteyn had announced his discovery that the stars could be separated into two large "streams" which were in motion relatively to each other. This result was based on an analysis of the motions of the brighter and nearer stars, but in 1906 A. S. Eddington analysed the motions of the Groombridge stars. This investigation confirmed the phenomenon of star-streaming, and extended it to the fainter stars.

In December 1905, subsequent to the death of Professor Ralph Copeland, Dyson was appointed to the conjoint posts of Astronomer Royal for Scotland and Professor of Astronomy in the University of Edinburgh. At the time of his appointment the Royal Observatory, Edinburgh, was engaged on a programme of meridian observations of a list of zodiacal and other stars prepared by Sir David Gill. This programme was finished under the supervision of Dyson, who personally devoted a great deal of time to the work of reduction, and the results were published in 1910 as Volume III of the *Annals of the Royal Observatory, Edinburgh*. Spectroscopic work on the Sun's rotation was also

continued by J. Halm and subsequently by J. Storey, under Dyson's direction, and in 1908 a new programme of double star work was initiated which was continued up to 1915. An important addition to the work of the observatory consisted in the scheme of co-operation with Perth Observatory, Western Australia, whereby the plates for a large section of the Perth Astrographic Zone were measured and reduced at Edinburgh. Dyson threw himself into the supervision of this work with characteristic energy and enthusiasm.

During his tenure of office in Edinburgh Dyson published an important paper confirming the phenomenon of star streaming (see above), which had been received with cautious scepticism by many astronomers. It was felt that the results obtained by Kapteyn and others might be due to errors in the proper motions, but in 1908 Dyson completed an analysis, restricted to the stars of large proper motions, for which the possible errors amounted to a small fraction of the whole. This analysis showed decisively the reality of star-streaming. It was published in the *Proceedings of the Royal Society of Edinburgh* (vol. xxvii, p. 131), and this paper is now regarded as one of the classics on the subject of stellar motions. Another paper which analysed stellar motions in a rather different way, and which again confirmed the existence of a preferential direction of motion for the stars (apart from systematic motions due to the motion of the solar system) was published in March 1910 in the *Monthly Notices of the Royal Astronomical Society*.

On October 1, 1910, Dyson returned to Greenwich as Astronomer Royal, being succeeded at Edinburgh by R. A. Sampson. For the next 23 years he guided the fortunes of Greenwich wisely and well. Whilst meridian work was pushed forward with unabated vigour other programmes were successfully undertaken. Our knowledge of the stars in the Greenwich zone of the Astrographic Catalogue has been enriched by the successful prosecution of the Greenwich parallax programme. The whole zone was re-photographed with the astrographic telescope, and proper motions derived for stars down to the 13th magnitude. Determinations of photographic magnitudes in the zone were undertaken. The magnetic work of the observatory was developed and solar work prosecuted with unabated zeal. Double star work was continued and a programme of colour temperature observations initiated. An important branch of geophysical work which Dyson fostered was the determination of latitude variation. He constantly endeavoured to improve the accuracy of determinations of time. And even in an abbreviated notice reference must be made to his eclipse activities and to the fact that it was due to him that the two expeditions were sent to photograph the total solar eclipse of May 29, 1919. He had noticed in 1917 that at this eclipse

the sun would be in almost the most favourable possible position for testing the deflection of light predicted by Einstein's General Theory of Relativity, and in spite of the discouraging conditions of 1918 preparations for the eclipse proceeded. The confirmation of the Relativity Theory secured by the 1919 expeditions is now a matter of history, and it was due to Dyson that they were able to set out at all.

Essentially a team-worker, Dyson proved himself to be an ideal leader of an astronomical team. To a large extent he encouraged his subordinates to act on their own initiative, but they constantly consulted him on all matters as they realised that his advice was invaluable. He knew how to command the affection as well as the respect of those whom he led. One of the Greenwich staff once remarked, "all that our chief asks is that we do our best," and it was practically impossible for anyone not to do his best if he was fortunate enough to work under Dyson.

Towards the close of Dyson's tenure of office at Greenwich a fine 36-inch reflecting telescope, with its building and accessories, was presented to the observatory by Mr William Johnston Yapp, who stated that he made this gift specifically to commemorate Sir Frank Dyson's services as Astronomer Royal. At about the same time, as a result of Dyson's representations, the Admiralty sanctioned the construction of a new reversible transit circle which would replace the existing instrument. The acquisition of these instruments at the close of Dyson's official career has been of great value to Greenwich.

Subsequent to his retirement in 1933 Dyson lived in the neighbourhood of Greenwich Park, and his advice was much in demand and always available to his astronomical friends. He had married Caroline Bisset Best, daughter of Mr Palemon Best, M.B., and there were two sons and six daughters of the marriage. Astronomers from all parts of the world, in addition to Dyson's immediate colleagues, were always sure of a warm welcome in their home. Lady Dyson died in 1937 and Sir Frank followed her on May 25, 1939, his death occurring whilst on the return voyage from a visit to Australia.

Dyson received many honours. Among these may be mentioned a Royal Medal of the Royal Society in 1921 and the Bruce Gold Medal of the Astronomical Society of the Pacific in 1922, in addition to the Gold Medal of the Royal Astronomical Society which was awarded to him in 1925. He was created a Knight Bachelor in 1915 and a K.B.E. in 1926. He received honorary degrees from numerous universities, and was foreign or corresponding member of various national academies. He was elected a Fellow of the Royal Society of Edinburgh in 1906.

See also *Obituary Notices of Fellows of the Royal Society*, vol. iii, No. 8, 1940, pp.159–172.

W. M. H. G.

Major Sir Frederick T. G. Hobday, C.M.G., F.R.C.V.S.

THE death of Sir Frederick Hobday at Droitwich on June 24, 1939, has robbed the wide circle of his friends of a man of outstanding personality and unusual energy and ability.

Born at Burton-on-Trent, and educated at the Burton Grammar School, a love of animals which persisted throughout a long and active life turned Frederick Hobday's thoughts away from a business career to the study of veterinary medicine and surgery. He qualified at the Royal Veterinary College, London, in 1892, in the same year that Sir John M'Fadyean was appointed Dean. His early work was largely concerned with general practice. Later, he devoted much time and attention to a study of anæsthesia in animals and to veterinary surgery. He rapidly achieved a reputation as a surgeon, particularly perhaps as a specialist in abdominal surgery. Equine cryptorchidectomy, ovariotomy, and the operation for ablation of the laryngeal ventricles became associated with his name in Britain, but his services as a general consultant surgeon were widely sought.

During the Great War (1914–1918) he served in the R.A.V.C. in France and later on the Italian Front and in Albania. His able operative ability and his efficient war services were recognised by numerous awards. He was twice mentioned in despatches, was made a Companion of St Michael and St George, and received numerous foreign decorations and awards.

After the Armistice he spent a period in veterinary practice in London, and in 1927 was elected to succeed Sir John M'Fadyean as Principal and Dean of the London Veterinary College.

It was while holding this post that he carried out the work for which he will be best remembered. He found the college buildings in a state of great neglect, the finances of the institution were lamentably inadequate, and the general morale was low. Stimulated by the disgraceful neglect and imminent disaster to what should have been the foremost veterinary teaching centre in Britain—the blame for which must rest with the State—he initiated a campaign to collect funds for complete rebuilding. In this he was so successful that the subscription fund reached £135,000, which, together with a sum of £150,000 from the Government, enabled

the whole of the old structure to be replaced by a building designed and equipped in accordance with modern ideas. He had the satisfaction of seeing the buildings completed, equipped, occupied, and opened by Their Majesties the King and Queen in 1937, before he relinquished the Principalship of the College.

During his lifetime Hobday did much to establish the art and science of veterinary medicine and surgery as a recognised profession. He began his career at a time when horse transport was of vital importance, and lived to see the gradual transition to mechanical transport. As equine work became less he devoted more time to a study of diseases of the smaller animals, and during his latter years he was perhaps better known as a canine and feline specialist.

He contributed to many periodicals, was editor of the *Veterinary Journal*, and published a number of textbooks.

He will be remembered as a man of immense mental and physical energy, who combined a dexterous manipulative surgical skill with an acutely humanitarian outlook. He was possessed of a natural kindly and charming manner, and invariably was ready to help the student in his difficulties.

He was elected a Fellow in 1904.

Wm. C. M.

James Gall Inglis.

Mr Inglis was born in Edinburgh in 1865. He was the son of Robert Inglis, head of the publishing firm of Gall & Inglis. On leaving school he had wished to continue the study of science, but at the death of his father he felt it his duty to enter the firm rather than proceed to the University. He managed the head office of Gall & Inglis from 1888 to 1900 and subsequently edited many of its publications, for example its numerous business reckoners. He also interested himself in the engineering side of the firm's printing works, the efficiency of which he greatly improved.

His chief hobbies were mountaineering and astronomy. He was a Vice-President of the Scottish Mountaineering Club, to whose *Journal* he was a frequent contributor, and spent many holidays, summer and winter, in the Scottish hills. Munro's Tables owe their last revision to his efforts. His best-known service to astronomy was the publication of *Norton's Star Atlas*, the notes comprising the first part of which work are largely his own. He had a flair for discovering points of interest about which little information is available. He was one of the earliest members and a past President of the Edinburgh Astronomical Society, drawing up its first constitution and acting for many years as Honorary Librarian. He was a Fellow of the Royal Astronomical Society of London.

A sincere Christian, he devoted much time and energy to working for his Church. He was the senior elder and treasurer of the Foreign Mission Fund of Mayfield North Church, and taught in its Sunday School for some fifty years. He leaves a widow and one son, Mr R. M. Gall Inglis, who besides succeeding him in the management of the firm of Gall & Inglis inherits many of his father's interests.

He was elected a Fellow in 1920, and died on April 6, 1939.

E. A. B.

Colonel Henry Halcro Johnston, C.B., C.B.E., D.L., D.Sc., M.D., C.M., F.L.S.

HENRY HALCRO JOHNSTON, the fifth son of James Johnston of Coubister, Orphir, Orkney, was born on September 13, 1856, and died on October 18, 1939. He received his early education at Dollar Academy and at the Edinburgh Collegiate School. Entering the University of Edinburgh he completed his medical course and subsequently attended the Army Medical School at Netley. He joined the Army Medical Department in 1881. As a student at Edinburgh he was particularly prominent as an athlete and represented his University at various sports. Tall in stature and with a powerful physique he was eminent as a hammer thrower and took his place on the rugby international field for Scotland against England and Ireland in 1877. His first service overseas was in 1885 at Suakin. He then spent the years 1887–90 at Mauritius, and after transference to India took part in the Malakand and Buner Expeditions in the North-West Frontier during the years 1897–98. For service in South Africa, 1899–1902, he was awarded the honour C.B. He was promoted Colonel in 1911 and retired two years later, but almost immediately he was on duty again from 1914–19 as Deputy Director of Medical Services at Gibraltar and subsequently as Assistant Director of Medical Services for the Western District of Scotland and for the Northern Command, England. At the conclusion of the war he received the honour C.B.E. On his final retirement he resided chiefly in Orkney and was a Deputy Lieutenant for the county.

Throughout his career abroad and in the midst of his normal duties Colonel Johnston was keenly interested in the flora of the various countries in which he campaigned, and took the opportunity of making collections of herbarium material as well as seeds. His interest was chiefly in these collections and not in the writing of papers. He, however, published from time to time botanical notes particularly on the flora of Mauritius and on the flora of the Orkney and Shetland Islands. Most of his contributions are to be found in the *Transactions of the Botanical Society of Edinburgh*. It was only in 1919 that he was able to give himself up almost wholly to botanical studies. His chief pursuit was a most careful scrutiny of the flora of the Orkney Islands. In addition to a most exact

enumeration of the present flora, he examined all earlier records and did his best either to confirm or to disprove such as were doubtful. His own knowledge of the vegetation of every one of the Orkney Islands was particularly complete and it is unlikely that any other county in the British Isles has been examined with the same amount of detail. He was scrupulously exact in his methods, and his notes on the flora are a model of accuracy and of care. His extensive collections from his native county were gifted by him to the custody of the Orkney Museum. It was his practice to submit the majority of his specimens to careful scrutiny by experts in each family or genus, and his records consequently can be regarded as very dependable. The rest of his collections with the valuable notes attached were bequeathed by him to the Edinburgh Royal Botanic Garden.

Colonel Johnston retained his mental and bodily vigour until on the verge of his eightieth year, and it was only during the last three years that he was prevented by illness from continuing his personal expeditions into every island of the group. He was unmarried.

He was elected a Fellow in 1895.

W. W. S

Sir Robert Ludwig Mond, LL.D., F.R.S.

THE death of Sir Robert Ludwig Mond, which occurred in Paris on October 22, 1938, deprived this world of a man who was beloved by hosts of friends in many countries and who had devoted his life to the advancement and promotion of science, the well-being of humanity, and the cause of civilisation. At the International Congress of Chemistry held in Rome during May 1938, he was aptly described by Sir Robert Robinson as the "great ambassador of science and friendship amongst the nations."

He was born on September 9, 1867, at Farnworth, near Widnes in Lancashire, the eldest son of the famous chemist and industrialist, Dr Ludwig Mond, F.R.S., and was educated at Cheltenham College, St Peter's College, Cambridge, the Zürich Polytechnikum, and the Universities of Glasgow and Edinburgh. After this very thorough training, he was chosen by his father to assist him in his scientific researches and his great industrial undertakings. In 1897 he became a director of Brunner, Mond & Company. He rendered Dr Ludwig Mond invaluable assistance in the investigation of the metal carbonyls and the establishment of the famous Mond nickel process, becoming a Director, and later Chairman, of the Mond Nickel Company. Robert Mond deserves a special place in science for his own researches on the metal carbonyls, which amongst other things led to the discovery of cobalt nitrosyl carbonyl. These investigations and many others were continued and extended in the laboratory which he set up in his country house at Combe Bank, near Sevenoaks in Kent. Here he devoted great attention to the problems of scientific agriculture, and to the production of pure untainted milk, especially for young children.

In the course of his life Robert Mond became a famous archæologist, devoting many years to active and important exploration in Egypt and Palestine, and giving valuable financial aid, direction, and inspiration to the work of many others in this field of discovery. It was doubtless the importance of his archæological investigations which led to his receiving the high honour of election as a *Membre de l'Institut* (*Académie des Inscriptions*).

The Davy-Faraday Laboratory of the Royal Institution, which

Dr Ludwig Mond founded and endowed in 1896, owed very much to Robert Mond, who designed the fittings and installation and selected the initial equipment of apparatus. In later years he gave large sums, amounting to many thousands of pounds, to the improvement of the Royal Institution and the work of the Davy-Faraday Laboratory.

Many other important scientific Institutions greatly benefited by his deep interest in the promotion of scientific investigation and his splendid generosity. Thus he took a prominent part in the establishment of the Norman Lockyer Observatory at Sidmouth, and contributed largely to its maintenance, whilst the great *Maison de la Chimie* in Paris received from him a gift of a million francs. It was through his personal efforts and his generous benefactions and guarantees that the British Academy and the Palestine Exploration Fund were induced to join in establishing a "British School of Archæology in Jerusalem."

It was, however, not only institutions for science and scientific research which claimed his attention and his liberal support. His deep interest in the welfare of humanity and his kind and generous heart led to countless gifts to persons and institutions which no one but himself could have fully enumerated. A very notable example of this side of his life's work was his foundation of the Hospital for Infants at Vincent Square, as a memorial to his first wife. This Hospital received generous support from him throughout his life, and was for many years supplied with pure milk from his selected herd at Combe Bank.

Robert Mond received the honour of knighthood in 1932, and was promoted *Commandeur de la Légion d'Honneur* from that of *Officier*. The Universities of Liverpool and Toronto conferred on him the honorary degree of LL.D., whilst in the last year of his life he was elected a Fellow of the Royal Society—a long-deferred honour which gave him special pleasure. He was an original member of the Kaiser Wilhelm Gesellschaft zur Förderung der Wissenschaften. For many years he was the Treasurer and never-failing friend of the Faraday Society. During the period 1930–32 he was President, and by his deep interest in chemical science, his kindly tact, and his knowledge of men and affairs, greatly added to the prosperity, good fellowship, and international prestige of the Society. Few of our friends from overseas will ever forget the memorable Manchester meeting of September 1932 and the princely generosity of the President.

In 1898 he married Helen Edith, third daughter of the late Mr Julius Levis. She died in 1905. The two daughters of this marriage survive him, René (Mrs James Dunn) and Frida (Mrs H. G. Brackley). He married in 1922 Marie Louise, daughter of the late Mr G. J. le Manach,

of Belle-Isle-en-Terre, Brittany. Lady Mond, to whose loving care Sir Robert owed so much in his later years, now lives mostly in France.

After giving up Combe Bank, Robert Mond lived, whilst in London, at 9 Cavendish Square, where he had a wonderful collection of objects of art and antiquity, and a great room which was a fine reproduction of an apartment in a Pharaoh's palace. In the later years of his life his many friends were often hospitably entertained at his beautiful villa at Dinard or his chateau in West Brittany.

Robert Mond was one of those great men of Jewish descent who have not only adorned and enriched English life but have promoted the highest ideals of human friendship, culture, and civilisation throughout the world. A firm believer and an active worker in the progress of humanity through the discoveries and wise applications of modern science, he was also a lover of the beautiful in Art and an earnest and successful investigator of man's achievements in ancient civilisations. These words are true and deserve to be inscribed on the incorruptible tablets of history, but what shall we personally say, his many friends who live to mourn the loss of that fine, generous, and ever-youthful spirit? It is hard indeed to describe such a beloved friend to those who did not know him, for no ordinary crystallisation of thought in words can reveal the personal quality of a great-hearted human soul. Let us humbly say that Robert Mond lives in our hearts, a dearly cherished and very precious memory.

He was elected a Fellow in 1890.

Note.—A very good Obituary Notice of Sir Robert Mond appeared in *The Times* of October 24, 1938, and in a contribution to the same Journal on November 3, 1938, Professor S. R. K. Glanville gave a valuable account of his fine work as President of the Egypt Exploration Society. In the *Obituary Notices of Fellows of the Royal Society*, published in the *Proceedings* of January 1939, there appeared an excellent memoir from the pen of Sir Jocelyn Thorpe, which contains extracts from the funeral orations pronounced in Paris by MM. Auguste Béhal and Louis Hauzeur.

F. G. D.

John Murdoch Murray, B.Sc.

JOHN MURDOCH MURRAY died on September 30, 1939, very unexpectedly at the early age of fifty-two. He went to the University of Edinburgh from his country home in Perthshire uncertain whether to make agriculture or forestry his career. Fortunately for the latter he determined upon it. He graduated in Agriculture in March 1914 and in Forestry in the following year. He took first place in both advanced and practical Forestry at the University and generally stood well in his agriculture classes. He joined me in the War Exploitation of home woods in 1915–16—a duty little appreciated by either of us—and when the Forestry Commission was formed in 1919 he became a junior District Officer, and at his death was in charge of the whole Scottish Forestry Organisation.

As a boy he gained a knowledge of field work in the woods and on the farms. He was a keen observer and combined in himself sound scientific principles with a practical sense. It fell to him to direct and instruct young foresters and he took much pleasure in helping them in many ways.

As Assistant Commissioner for Scotland, Murray had under his supervision many Officers and Foresters and a few thousand men. His treatment to all was fair, and where rebuke was necessary it was given so quietly as to make the culprit feel doubly sensible of his guilt. Forestry profited through Murray in many directions. Large areas of forest were designed and planted by him with unerring judgment of soils and conditions. It is difficult to select the best forester of this century, but in my estimation Murray would certainly have been one on a very short leet. The Forestry Commission has decided to name a plantation at Glentress near Peebles to his memory, as an appreciation to his outstanding services. Murray made a very exhaustive study of the Scots Pine, travelling all over Scotland as well as on the Continent upon this investigation. It is to be hoped that the result will not be lost and that his copious notes will later be made available to students of sylviculture.

He was elected a Fellow in 1932.

J. S.

The Very Rev. William Paterson Paterson, D.D., LL.D.

By the sudden passing on January 10, 1939, of the Very Rev. Professor Paterson, Scotland has lost a highly gifted son and learning a brilliant luminary. Born at Skirling in 1860, he received his early education at the parish school and the Royal High School, Edinburgh. A notable graduate in Arts and Divinity of the University of Edinburgh, he completed his theological studies at the Universities of Leipzig, Erlangen, and Berlin. From 1887 to 1894 he was parish minister of Crieff. In the latter year he was appointed to the Chair of Systematic Theology at the University of Aberdeen—an office which was then filled by examination (recently abolished), from which he emerged as the successful competitor. In 1903 he was elected to succeed Professor Flint, distinguished alike as theologian and philosophical historian. In 1916 he was appointed Chaplain to the King in Scotland, and three years later was elected Moderator of the General Assembly of the Church of Scotland. As Dean of the Faculty of Divinity, he took a leading part in the reorganisation of the Courses for the B.D. degree of the University of Edinburgh and in the inauguration of that of Ph.D. in Theology for the purpose of stimulating specialisation and research in theological study. His early reputation as a theologian is evinced by his appointment as Baird Lecturer in 1903—the lectures being subsequently published under the title *The Rule of Faith.* A still greater testimony to his reputation as a theologian was his selection as Gifford Lecturer at the University of Glasgow in 1923, the fruit of which appeared in his masterly work, *The Nature of Religion.* In 1931 he was invited to deliver the Sprunt Lectures at Richmond, Virginia, and four years later the Forman Lectures at Liverpool, while in 1912 he filled, by invitation, for six months the pulpit of the Scots Church at Melbourne. The far-flung eminence, to which these appointments testify, also brought him many academic honours. Edinburgh, St Andrews, and Trinity College, Dublin, conferred on him the Hon. D.D. degree; Edinburgh, Glasgow, and Pennsylvania that of LL.D.

Paterson was a man of keen, penetrative mind and of wide culture. His retentive memory enabled him to work with ease his wide knowledge into his lectures, books, sermons, and speeches. He was a lucid and effective teacher, who drew students from many lands to his classroom.

The essentials of a subject were set forth with a rare precision, which shunned verbosity, and yet was never dull. His nimble mind was always working, point by point, towards a demonstration of the problems under discussion. He was likewise a telling speaker, in faculty and senate, on the public platform, and on social occasions. His ready wit lightened the subject when the occasion permitted. Like his culture, his interest and his activity took a wide sweep. He was an ardent sociologist and social worker. Witness his activity on behalf of the temperance movement and his collaboration with the Rev. Dr D. Watson of Glasgow in a volume on *Social Evils and Problems*. Many causes in behalf of human betterment could count on his energetic support in committee, on the platform, and in the press. In this wider sense his death is a grevious loss to his country as well as to the Church, while it has deprived his friends and colleagues of one whose rich personality and kindly tolerance ever attracted and endeared.

He was elected a Fellow in 1918.

J. MACK.

Sir Robert William Philip, M.A., M.D., LL.D., F.R.C.P.E.

ROBERT WILLIAM PHILIP, who died on January 25, 1939, was born at Govan on December 29, 1857. His father, the Rev. George Philip, D.D., was later called to a charge in Edinburgh, and there Philip received his early education at the High School and University, where he took a degree in Arts before entering on the study of medicine. In 1882 he graduated M.B., C.M., with honours, then spent some further time in post-graduate study at certain of the great Continental schools, and served for a period as house-physician in the Royal Infirmary. Shortly thereafter he was appointed Assistant to the Professor of Medicine in the University and started in private medical practice. His appointment gave him an introduction to teaching besides affording opportunity for research work, and on his promotion to M.D. in 1887 his graduation thesis was awarded a gold medal and the Gregory Prize. In 1890 he was elected to the honorary medical staff of the Royal Infirmary, to which he gave life-long service as physician and clinical teacher. During this time he became an independent lecturer in the Medical School, devoting his teaching to general medicine and largely, but by no means exclusively, to diseases of the chest. He also associated himself with the affairs and activities of the more important medical associations and took a prominent position in their discussions. Koch's discovery of the tubercle bacillus in 1882 had, however, made a profound impression on Philip, upsetting the tenets in which he had been educated, and its corollary of the unity and infectivity of all the various manifestations of tubercular disease had opened up a completely new conception of the whole subject. Much thinking over it led him to conclusions which determined the principal activities of his life. At that time in this country tubercle exacted a higher death rate than any other single disease, its victims were drawn largely from the poorer section of the people, and its chronicity kept the number of sufferers at a very high level. The theory of its hereditary origin still held sway, little importance was attached to social conditions as a factor in causation, its prevalence was regarded as a matter of course, and no public provision existed for dealing with it. To abolish an evil of this magnitude was a stupendous task involving years of missionary and constructive work which he started in 1887 by establishing "The

Victoria Dispensary for Consumption." Philip's survey of the whole situation had led him to the conclusion that the great source of infection and spread lay in the homes of the people and that early cases must be sought out there and suitably cared for and supervised as long as necessary. To accomplish this the "Dispensary" was planned and conducted on an entirely novel basis as the co-ördinating centre of all the activities of the campaign, and to it were added as funds allowed the other elements of the scheme—sanatorium, farm colony, hospital for advanced cases, nursing, home supervision, charitable help, suitable employment, and so on. The organisation is known throughout the world as the "Edinburgh Tuberculosis Scheme" and has been adopted by all the more progressive countries. Its inception, development, and success stamp its author as a great physician and an outstanding benefactor of his time.

In recognition of his public services he received the honour of knighthood in 1913 and the appointment of Honorary Physician to the King in Scotland. In 1917 he was elected Professor of Tuberculosis in the University of Edinburgh and in 1932 Chairman of the National Association for the Prevention of Tuberculosis. He was President of the Royal College of Physicians of Edinburgh (1918–23) and of the British Medical Association (1927), besides being the recipient of many honorary degrees, fellowships and other distinctions from universities, colleges, and medical societies at home and abroad. He became a Fellow of the Royal Society of Edinburgh in 1889 and served as Vice-President from 1927–30. Apart from the medical side of his life Philip's personality was of unusual interest—hospitable, well read, widely travelled, a collector and connoisseur of objects of art, an interesting talker, and an effective public speaker almost equally fluent in English, French, or German.

R. S.

Thomas Stephenson, D.Sc., F.C.S.

BY the death of Dr Stephenson on October 29, 1938, there passed a distinguished and in some ways unique figure in the world of Pharmacy. He was the grandson of a doctor and the son of a chemist, and probably as the result of his early associations he displayed while quite a young man some of those traits which characterised his maturer years.

He was born in 1864, and became a qualified chemist in 1886, and continued the habit, which he had formed as a student, of reading papers at various professional meetings. He read widely and had the gift of grasping the fundamentals of the subjects, which came under his review.

For a time he held a position in Bombay, and his inquiring mind led him to obtain a knowledge of the various indigenous remedies, a knowledge which was always at the disposal of others. This gift of acquiring and imparting knowledge was a part of his natural make-up, and he was always able in conversation to throw many interesting sidelights on the lands and peoples he had visited in his much travelled life.

He was deeply interested in his work especially on its scientific side, and this found expression in more ways than one.

While engaged in pharmaceutical practice he felt a growing desire to relinquish Pharmacy for a wider sphere of scientific writing, for which he had a distinct flair.

The study of drugs and their reaction on each other and on the body was of absorbing interest to him, and thirty-one years ago he launched *The Prescriber*, a monthly periodical which became an almost universally read epitome of all that was new in medicines. This journalistic venture was a complete success.

Dr Stephenson's room was typical of the man, neat and orderly, its walls surrounded by books whose titles revealed something of his love for the arts, especially Music and Literature, because his tastes carried him far beyond the limits of merely professional work. He found considerable delight in History, the great Novelists, Music and Opera, and especially the Gilbert and Sullivan Operas fascinated him and on the latter he was quite an authority. But he had still room in his heart for his fellow-men, and his devotion to the Rotary Movement was one of the outward tokens of his feelings.

We remember Dr Stephenson as we last saw him, youthful in figure and dignified in bearing, mentally alert and always busy, and it is not easy to realise that this courteous, polite and well-informed gentleman is with us no more.

He was elected a Fellow in 1910. J. O.

Arthur Logan Turner, M.D., LL.D., F.R.C.S.E.

THE "passing" on June 6, 1939, of Arthur Logan Turner will be deeply regretted by all those who, irrespective of their nationality, have at heart the best interests of Octology, Laryngology, and Rhinology. Coincidently, they will sympathise with the University of Edinburgh, which has lost one of its most distinguished sons.

In the limited space at my disposal, it must suffice to envisage the personality of the man and the nature and quality of his work as I believe them to have been estimated by his confréres south of the Border.

It is not improbable that a stranger meeting Logan Turner for the first time might have found him reserved and somewhat difficult to approach. But as so often happens in like circumstances, further acquaintance soon proved those features to be a thin armoury which sheltered a kind, very modest, and always sympathetic friend.

Unless my memory be at fault, it was in 1893 when he and I found ourselves amongst the junior members of the recently established "Laryngological Society of London" which, in its ambit, included Rhinology. After a few of its monthly meetings, it became evident that we were particularly interested in inflammatory affections of the frontal sinuses and their possible, local or distant complications. It was a subject which, hitherto, had received little attention from rhinologists because the upper air-cells represented almost unexplored regions of the skull and treatment (*sic*) of pathological discharge from them was (according to a senior American surgeon) confined to intra-nasal "snipping and spraying—mostly sprays." Hence, and with his usual perspicacity, Logan Turner realised that without a full knowledge of the anatomy of the sinuses, it would be impossible to realise the significance of their clinical manifestations, or to base treatment of them on the ordinary principles of surgery.

With a wealth of skulls available, he was soon able to determine the normal size and configuration of the air-cells, their interrelationships and commoner variations. Meanwhile, his clinic in the Royal Infirmary afforded ample opportunities for determining to what extent symptoms and signs might be modified by the structural factors.

These combined observations, with a number of illustrations, were

published in October 1901 and entitled *The Accessory Sinuses of the Nose, their Surgical Anatomy and the Diagnosis and Treatment of their Inflammatory Affections*.

In 1923 he was chosen to deliver the annual "Semon Lecture" (University of London) and, as a born teacher, his confréres were not surprised that he took as his subject "The Advancement of Laryngology and Octology. A Plea for Adequate Training and Closer Co-operation."

With these and the many other services he rendered to our special departments of Medicine, it is not surprising that American and Continental Societies enrolled our friend's name on their lists of Honorary Fellows.

It has always seemed to me that he was one of the few whose contributions on any disease he had studied betrayed an exceptionally ordered mind which enabled him to express his views with a clarity which many of us must have envied.

If, therefore, we bear in mind his personal qualities and the debt our speciality and the Edinburgh School of Medicine owe to him, it would not be extravagant to say of Logan Turner—

> "His life was gentle and the elements
> So mixed in him, that Nature might stand up
> And say to all the world, 'This was a man!'"

He was elected a Fellow in 1905, served on the Council from 1926 to 1929, and as a Vice-President from 1930 to 1933.

H. T.

Robert Wallace, M.A., LL.D., F.L.S.

DR ROBERT WALLACE, Professor of Agriculture and Rural Economy in the University of Edinburgh from 1885 to 1922, and Garton Lecturer on Colonial and Indian Agriculture from 1900 to 1922, died on January 17, 1939, at the age of eighty-five at Mid Park House, Kincardine-on-Forth. He was a man of exceptional ability and great mental energy. For the thirty-seven years of his professorship he was recognised as the leading academic representative of agriculture in Scotland. Reared on the land, he inherited a love of the soil. Members of his family have been farming in Dumfriesshire and Galloway for nearly a century and a half. His father, the late Mr Samuel Wallace, Wallace Hall, Glencairn, was one of the largest and best-known farmers in the south of Scotland.

The future Professor received his early education at Tynron School and Hutton Academy. He proceeded later to the University of Edinburgh, where he studied under Professor Wilson, who held the Chair of Agriculture. After a brilliant academic career at the University he returned to the land, and for several years managed farms for his father, and also farmed for a time on his own account along with his brother, Mr S. Williamson Wallace who, in 1902, became Director of Agriculture for the State of Victoria in Australia.

Professor Wallace began his career as a teacher of Agriculture in 1882, when he was appointed Professor of Agriculture for the Royal Agricultural College, Cirencester. Three years later he succeeded Professor Wilson as Professor of Agriculture and Rural Economy in the University of Edinburgh. Under his enlightened and energetic direction the curriculum for agricultural students was broadened and improved by the addition of a course in Forestry in 1885, and by a course of lectures in Agricultural Entomology in 1891. In 1900 a course of lectures on Colonial and Indian Agriculture, delivered by Professor Wallace himself, was added. In 1886 the University, acting on his advice, established the degree of B.Sc. in Agriculture. These developments greatly enhanced the status of the training given in agriculture: Edinburgh, in fact, became one of the leading schools of agriculture in the Empire.

Professor Wallace was a great traveller. His search for new knowledge regarding agricultural and pastoral conditions took him in 1887 to Italy and India; in 1890, 1892, and 1898 to the United States; in 1891 to Egypt; in 1891 and 1892 to Greece; in 1895 to South Africa;

in 1896 to Australia and New Zealand; in 1907 to Canada, the United States, and Mexico; and in 1908 to Canada, the United States, and Rhodesia. Year by year he added to his stock of knowledge, and freely gave it out again to those whom he reached with his never-resting pen. He was a prolific writer both of books and of articles and letters for the Press. His best-known book was perhaps *Farm Live-Stock of Great Britain*, which went through several editions. Other books of his deal with agricultural and pastoral conditions in India, in Australia and New Zealand, and in Cape Colony. Some of his books, such as *A Country Schoolmaster*, *Eleanor Ormerod, LL.D.*, *Heather and Moor Burning for Grouse and Sheep*, are less formal in character; but all bear the stamp of his alert and inquiring mind.

His empire-wide reputation brought him into touch with the governing authorities and commercial concerns in many parts of the Empire, and his services in an advisory capacity were much in demand. In 1879 he went to Canada to report on that Dominion as a field for the emigration of agriculturists. In 1895 he visited Cape Colony on the invitation of the Government of South Africa. In 1893 he revisited Canada as a Special Commissioner to report to the Government at Ottawa on Highland Crofter Settlements in Manitoba. For five years he was consulting adviser to the British South Africa (Chartered) Company.

From his students' point of view he was the perfect professor. His lectures were characterised by their freshness of outlook: there was nothing bookish about them; for they were largely based on his own varied farming experience and the investigations which he himself had made during his wide travels. In his students he took a paternal interest, and through his influence many of them got their first appointments. In their subsequent careers he continued to keep in touch with them despite the fact that he was a very busy man. In losing him many of us have lost a true friend.

Professor Wallace was an LL.D. of Edinburgh; a Fellow of the Royal Scottish Geographical Society, and of the Royal Physical Society of Edinburgh, and a member of the Highland and Agricultural Society of Scotland.

He was a great patriot who dearly loved his country. His *Letters to President Wilson*, written in the early years of the Great War, reveals the ardent patriotism of the man, and it may be that these letters served their purpose in influencing the President towards bringing America into the war.

He was elected a Fellow in 1886.

D. C.

Edmund Beecher Wilson, For. Mem. R.S., Hon. F.R.S.E.

In 1923, when Professor E. B. Wilson was elected an Honorary Fellow of this Society, he had already for a generation been a leader in the most rapidly developing of all the kin branches of Biology—the study of the cell. He saw the new science of cytology rise upon the discoveries of Boveri and Van Beneden, and the successive editions of his own work upon *The Cell* mark the stages of that phenomenal progress. The first edition, published in 1896, not long after these discoveries, guided attention to and focussed it upon the new development; the second, published in 1900, remained for a quarter of a century the handbook and inspiration of student and researcher.

And since the cell is the foundation of all living organisms his conceptions opened up fresh avenues of exploration in many directions. The discovery that at an early stage in the developing egg, a cell might retain the power of reproducing the whole organism, or might be set aside as the starting-point of a particular organ, led to the idea of cell-lineage in embryology.

It was he and his students who laid the foundation of modern notions of the inheritance of characters by interpreting Mendelian segregation as a matter of parental chromosomes kept distinct in the germ cells, and who later discovered the sex-determining chromosomes in certain insects.

These and the multitudinous discoveries of the first quarter of the present century, including the revolutionary work of a colleague of Wilson's in Columbia University, T. H. Morgan, followed the appearance of the second edition of *The Cell*, and demanded the production of a new summing up of the situation. So there appeared in 1925 a third edition, entirely rewritten and much bulkier, under the title *The Cell in Development and Heredity*, which another of his American colleagues, Professor E. G. Conklin, in presenting him for the Daniel Giraud Elliot Medal of the National Academy of Sciences, called "in every respect a monumental work, one of the most complete and perfect that American science has produced in any field."

Wilson made other important contributions to zoological knowledge, but his name will be most closely associated with the rise of the twin branches of cytology and genetics. He died on March 3, 1939, eighty-two years of age.

See also *Obituary Notices of Fellows of the Royal Society*, vol. iii, No. 8, 1940, pp. 123–138. J. R.

Sir Robert Patrick Wright, LL.D.

Sir Robert Wright may be regarded as in the line of succession to Henry Stephens who was Secretary and Treasurer of the Royal Society Club for sixteen years and wrote the classic *Book of the Farm.* Sir Robert in his turn edited the *Standard Cyclopedia of Modern Agriculture*, the most useful and comprehensive reference book on agriculture so far published.

The son of a farmer, and a farmer himself, Sir Robert's active mind soon took him beyond the confines of his farm. He had studied the agricultural science of the day at the University of Edinburgh and, debarred from a renewal of his lease by his opinions on land tenure, he set himself to organise agricultural education in the West of Scotland. Beginning with three pupils in the old Technical College, he lived to see the fruition of his efforts in the great College in Glasgow and in the most modern of dairy schools at Auchencruive. Appointed agricultural adviser to the Congested Districts Board, he became almost automatically the first Chairman of the Board of Agriculture.

In the stormy days of Land Settlement, when it was largely a political question, the constructive and lasting work of the Board under Sir Robert's direction was overlooked. The reorganisation of agricultural education until every farmer in every corner of Scotland was entitled to personal advice and instruction was part of his administration. The schemes for the improvement of Livestock were set up during his term, and but for the intervention of the War other valuable developments would have taken place.

Sir Robert was a pioneer. It was he who developed the method of field trials for the dual purpose of instruction and demonstration which, with modification, has been universally adopted. He was the leader of those who made the educational and research structure of Scottish farming second to none. He had the qualities of a leader, courage and persistence in his opinions, great enthusiasm, powers of persuasion, and devotion to public service. When temporarily defeated his custom was "reculer pour mieux sauter."

He was a voracious reader and, buried in a book, gave the Rocky Mountains merely a fleeting glance when passing through them, and he has been known to confess that he was more interested in English Literature than in Land Settlement.

He was a kind friend and a deeply religious man who remained a convinced "free" churchman; no compromise with a "State" church for him. All his life he was a devoted public servant and a worthy Fellow of the Royal Society to which he was elected in 1896.

He died on December 19, 1938. R. B. G.

WALTER JOHN MABBOTT, M.A., who was Rector of Berwickshire High School, Duns, from its inception in 1896 until he retired in 1930, died at his home at Haslemere, Surrey, on January 31, 1939, aged seventy-three. He was educated at Portsmouth Grammar School, the College of Science, Newcastle, and graduated with Honours in Mathematics at the University of Durham. For nine years, after leaving the University of Durham, he was Science Master at Merchiston Castle School, Edinburgh.

Mr Mabbott had a long connection with the Volunteers and Territorials, and held the rank of Major in the latter. He was Adjutant to the 4th Battalion K.O.S.B., and served with that unit during the war (1914–1918).

He was elected a Fellow in 1894.

TARAK NATH MAJUMDAR, L.M.S., D.P.H., D.T.M.(Cal.), F.C.S., Professor of Hygiene, Carmichael Medical College, Calcutta, since 1919, was also for many years City Health Officer for the Corporation of Calcutta.

He was elected a Fellow in 1913, and died on February 2, 1939.

LT.-COL. DAVID WATERS SUTHERLAND, C.I.E., M.D.(Edin.), F.R.C.P.(Lond.), I.M.S. (retired), was born in 1871 in Australia and educated at Melbourne and at the University of Edinburgh. He proceeded to India, was appointed Medical Officer to the Q.O. Corps of Guides, and took part in the Chitral Campaign, being mentioned in despatches and receiving the medal and clasp. In 1919 he was Consulting Physician to the Afghanistan Field Force. Lt.-Col. Sutherland held the Professorship of Pathology and Materia Medica at the Medical College of Lahore from 1897 to 1909, when he was appointed Professor of Medicine and Principal of the King Edward Medical College, Lahore. He was for some time Hon. Surgeon to the Viceroy of India. His publications include papers to the *Lancet* and *Brit. Med. Journ.*

He was elected a Fellow in 1903 and died on April 19, 1939.

ALEXANDER G. WALLACE, M.A., a native of Aberdeen, he was educated at the Free Church Training College Practising School, proceeding later to Edinburgh Free Church Training College and the University of Edinburgh. His first appointment after leaving school was at Invergordon Public School in 1882, and three years afterwards he returned to Edinburgh to graduate M.A. with Honours in Natural Science. He was Headmaster of the Central School, Aberdeen, from its opening in 1894 until he retired in 1926, and guided its development from an ex-standard School to a full Secondary School.

He was elected a Fellow in 1902, and died on December 27, 1938.

APPENDIX.

CONTENTS.

PROCEEDINGS OF THE STATUTORY GENERAL MEETING

Beginning the 156th Session, 1938–1939.

At the Statutory General Meeting, held in the Society's Rooms, 24 George Street, on Monday, October 24, 1938, at 4.30 P.M.

Sir D'Arcy Wentworth Thompson, President, in the Chair.

The Minutes of the Statutory Meeting held on October 25, 1937, were read, approved, and signed.

The President nominated as Scrutineers, for the election of Office-Bearers and Council, Mr WILLIAM FRASER and Mr F. C. MEARS.

The Ballot was then taken.

The General Secretary submitted the following report:—

GENERAL SECRETARY'S REPORT, OCTOBER 24, 1938.

During the session three Addresses have been given by request of the Council: on October 25, 1937, by Mr J. W. SLESSER MARR, on "Antarctic Surveys: the Work of the 'Discovery' Investigations"; on December 6, 1937, by Dr H. J. Bhabha, on "The Production of Electron Showers by Cosmic Rays"; and on February 7, 1938, by Dr G. W. TYRRELL, on "Geology in the U.S.S.R." On July 4, 1938, the Society celebrated, conjointly, with the members of the ST ANDREWS MATHEMATICAL COLLOQUIUM (July 4 to 15, 1938), the Tercentenary of the birth of JAMES GREGORY, the Scottish Mathematician. Certain papers on his life and work were read which will be published by the Society in a GREGORY MEMORIAL VOLUME, edited by Professor H. W. TURNBULL, F.R.S., and Dr A. C. AITKEN, F.R.S.

37 papers were read as compared with 33 in the previous session. The papers were on the following subjects: Mathematics and Statistics, 6; JAMES GREGORY papers, 5; Geology and Palæontology, 6; Botany, 4; Zoology, 8; Animal Genetics, 8. 13 papers have been published in the *Transactions* and 16 in the *Proceedings*. 5 papers on the life and work of JAMES GREGORY will be published in the Gregory Memorial Volume. 6 papers have been withdrawn, and several papers are still under the consideration of the Council.

The Society has lost by death 18 Fellows and 3 Honorary Fellows. 41 Fellows and 3 Foreign and 2 British Honorary Fellows were elected. 4 Fellows resigned and 3 Fellows were removed from the Roll.

Invitations were received and the Society was represented as follows on the occasions mentioned:—

1. Jubilee of MAURICE LUGEON, Lausanne, November 20, 1937. Engrossed letter sent.
2. 140th Anniversary of the Naturhistorische Gesellschaft zu Hannover, December 11, 1937. Letter sent.
3. Centenary of the Trevandrum Observatory, India, December 23, 1937. Letter sent.
4. 150th Anniversary of the Royal Botanic Garden, Calcutta, January 6, 1938. Letter sent.
5. 50th Anniversary of the Geographical Society of Finland, Helsingfors, January 22, 1938. Letter sent.
6. International Congress of Leprosy, Cairo, March 21, 1938. Letter sent.
7. Association Guillaume Budé, Strasbourg, April 19–24, 1938. Letter sent.
8. Centenary of the Societas Scientiarum Fennica, April 28–29, 1938. Letter sent.
9. Dedication of the Franklin Institute, Philadelphia, May 19–21, 1938. Principal Sir J. C. IRVINE, F.R.S.
10. 150th Anniversary of the Linnean Society of London, May 24–26, 1938. The PRESIDENT.
11. Lord High Commissioner's Levee at the Palace of Holyroodhouse, May 25, 1938. Professor F. A. E. CREW, Lt.-Col. A. G. MCKENDRICK, Dr JAMES WATT, and Dr C. H. O'DONOGHUE.
12. University of St Andrews, Commemoration of the Tercentenary of the birth of JAMES GREGORY, July 5, 1938. Dr A. C. AITKEN, F.R.S.

13. Congrès Internationale des Sciences Anthropologiques et Ethnographiques, Copenhagen, August 1–6, 1938. Professor V. GORDON CHILDE.
14. International Congress of Entomology, Berlin, August 15–20, 1938. Dr A. E. CAMERON.
15. 14th International Conference of the International Federation for Documentation, Oxford, September 21–26, 1938. Dr DOUGLAS GUTHRIE.
16. Astronomical Society of Edinburgh. Opening of Calton Hill Observatory Buildings, October 7, 1938. The PRESIDENT.

Early in the session a General Committee was formed, consisting of the Council of the Society and a number of leaders in science and education, to secure a portrait of the President—Sir D'ARCY WENTWORTH THOMPSON—to be hung in the Rooms of the Society. Mr DAVID S. EWART, A.R.S.A., was commissioned to paint the portrait. Dr JAMES WATT kindly consented to act as Treasurer of the Fund, and a sum of approximately £370, from which certain expenses fall to be deducted, was contributed by Fellows of the Society and others interested. The portrait was completed, and was presented to the Society, on behalf of the subscribers, by Professor F. A. E. CREW, at the Ordinary Meeting held on July 4, 1938. A smaller portrait will be presented to the President.

An inscription marking the birthplace of JAMES CLERK MAXWELL, Natural Philosopher, has been placed by the Society, with the approval of the Hon. LORD ROBERTSON, the owner, on the house at 14 India Street, Edinburgh. The lettering was designed by Mr L. C. EVETTS, A.R.C.A., King's College, Newcastle-upon-Tyne.

The PRESIDENT was appointed as the representative of the Society on the Meteorological Committee of the Meteorological Office, London.

A dinner was held in the Rooms of the Society on June 23, 1938, in honour of Professor MAX PLANCK, Hon.F.R.S.E., then on a visit to Scotland.

At the Ordinary Meeting on July 4, 1938, the KEITH PRIZE (1935–1937) was presented to Professor HAROLD STANLEY RUSE, M.A., for his paper "On the Geometry of Dirac's Equations and their Expression in Tensor Form," published in the *Proceedings* of the Society within the period, and for his other papers in the *Proceedings*; and the NEILL PRIZE (1935–1937) to Professor WILLIAM J. HAMILTON, M.D., D.Sc., for his contributions to the Embryology of the Ferret and other work published in the *Transactions* of the Society.

The DR W. S. BRUCE MEMORIAL PRIZE (1938) was awarded by the Joint Committee to Mr ALEXANDER R. GLEN, for his work in Spitsbergen, including survey in New Friesland and the completion of the map of North East Land. The Prize will be presented on December 5, 1938.

The JAMES SCOTT LECTURE (1938), on a subject dealing with the fundamental concepts of Natural Philosophy, will be delivered by Professor PAUL ADRIEN MAURICE DIRAC, F.R.S., on February 6, 1939.

The DAVID ANDERSON-BERRY PRIZE (1938) was awarded to MARY A. C. COWELL, M.B., Holt Radium Institute, Manchester, for her essay entitled "An Investigation into some of the Factors affecting the Response of Human Skin and Human Skin Tumours to Radiation." The Prize will be presented on December 5, 1938.

The sum of £300 was expended on Library binding during the session. Arrears in this Department are being gradually overtaken.

The thanks of the Society are due to those who have made presentations to the Library. These include a valuable collection of old biological volumes presented by Mr WILLIAM WILLIAMSON, a Fellow of the Society.

The acknowledgments of the Society are due to the Carnegie Trust for the Universities of Scotland for grants to authors towards the cost of illustrations, tabular matter, etc., of papers published in the *Transactions* and *Proceedings*, amounting to £203, 19s. 3d.; to Birkbeck College Publications Fund for a grant of £10 towards the publication of Mr A. GRAHAM's paper in *Transactions* (1938); to Professor W. T. GORDON, D.Sc., for £20 towards the plates of his paper in *Transactions* (1938); and for £300 received from the Royal Society of London, from the Government Publication Grant, in aid of the cost of the Society's publications for the session 1937–1938.

TREASURER'S REPORT:—

The TREASURER, in submitting the Accounts for the past year in a new form, mentioned the leading items of Receipts and Expenditure and compared them with those of the previous year. He stated that a sum of £200 had been set aside in the Accounts towards the cost of printing the GREGORY MEMORIAL VOLUME, and indicated that the financial position of the Society was satisfactory.

The Rt. Hon. LORD SALVESEN moved the adoption of the reports, and the reappointment of Messrs LINDSAY, JAMIESON & HALDANE, C.A., as auditors for the ensuing session. The meeting approved.

The Scrutineers reported that the Ballot Papers were in order, and the President declared that the following Office-Bearers and Members of Council had been duly elected:—

Sir D'ARCY W. THOMPSON, Kt., C.B., D.Litt., Hon.D.Sc., LL.D., F.R.S., President.

Professor F. A. E. CREW, M.D., D.Sc., Ph.D.
Lt.-Col. A. G. MCKENDRICK, M.B., D.Sc., F.R.C.P.E.
Principal J. C. SMAIL, O.B.E., Companion Inst.E.E.
Professor J. WALTON, M.A., D.Sc.
JAMES WATT, W.S., LL.D.
Professor E. T. WHITTAKER, M.A., Hon.Sc.D., LL.D., F.R.S.
} Vice-Presidents.

Professor JAMES P. KENDALL, M.A., D.Sc., F.R.S., General Secretary.

ALEXANDER C. AITKEN, M.A., D.Sc., F.R.S.
CHARLES H. O'DONOGHUE, D.Sc.
} Secretaries to Ordinary Meetings.

E. MACLAGAN WEDDERBURN, O.B.E., LL.D., D.K.S., Treasurer.

LEONARD DOBBIN, Ph.D., Curator of Library and Museum.

MEMBERS OF COUNCIL.

JOHN E. MACKENZIE, D.Sc.
Professor SYDNEY SMITH, M.D., F.R.C.P., D.P.H.
Emeritus Professor RALPH STOCKMAN, M.D., LL.D., F.R.C.P.E.
Professor LANCELOT T. HOGBEN, M.A., D.Sc., F.R.S.
Professor JAMES RITCHIE, M.A., D.Sc.
G. W. TYRRELL, A.R.C.S., D.Sc., F.G.S.
Emeritus Professor C. T. R. WILSON, C.H., M.A., D.Sc., LL.D., F.R.S.
Professor R. C. GARRY, M.B., Ch.B., D.Sc.
Professor R. J. D. GRAHAM, M.A., D.Sc.
Professor D. MURRAY LYON, M.D., F.R.C.P.E., D.P.H., D.Sc.
J. E. RICHEY, M.C., B.A., Sc.D., F.R.S., F.G.S.
The Hon. LORD ROBERTSON.

The Scrutineers were thanked for their services.

The PRESIDENT thanked the retiring Members of Council for their services.

The PRESIDENT referred to the loss the Society had sustained by the deaths of 18 Ordinary and 3 Honorary Fellows, including several distinguished and old Fellows of the Society.

Visitors having been admitted the PRESIDENT called upon Mr J. L. BAIRD, Hon.F.R.S.E., to deliver his Address on "The Development of Television."

The PRESIDENT, in name of the Society, cordially thanked Mr BAIRD for his Address.

PROCEEDINGS OF THE ORDINARY MEETINGS, Session 1938–39.

FIRST ORDINARY MEETING.

Monday, November 7, 1938.

Sir D'Arcy Wentworth Thompson, President, in the Chair.

The Minutes of the previous Ordinary Meeting were taken as read, and signed.

Dr ANDREW TOPPING signed the Roll and was admitted to Fellowship.

The following Papers were submitted:—

1. The Establishment of the Trichromatic Theory of Colour Vision. By Professor W. PEDDIE, D.Sc. *Proc.*, vol. 59, pp. 15–21.
2. The Cytology of the Thelytokously Parthenogenetic Saw-Fly *Thrinax macula* Kl. By Professor A. D. PEACOCK, D.Sc., and ANN R. SANDERSON, Ph.D. *Trans.*, vol. 59, pp. 647–660.
3. Some Eocene Ostracoda from North-West India. By MARY H. LATHAM, M.A. Communicated by Lt.-Col. L. M. DAVIES, M.A., F.G.S. *Proc.*, vol. 59, pp. 38–48.

Read by Title:—

4. Importance of Dialysis in the Study of Colloids. Part V: Colloidal Gold. Part VI: Colloidal Vanadium Pentoxide. By B. N. DESAI, B.A., LL.B., Ph.D., D.Sc., P. M. BARVE, M.Sc., and Y. S. PARANJPE, M.Sc. *Proc.*, vol. 59, pp. 22–37.
5. The Molecular Spectra of the Hydrogen Isotopes. I: Application of the Rotating Vibrator Model to the States of D_2. By IAN SANDEMAN, D.Sc. *Proc.*, vol. 59, pp. 1–14.
6. The Computation of the Error Function. By Professor G. N. WATSON, F.R.S. Communicated by Professor E. T. WHITTAKER, F.R.S.
7. Solution in Multiple Series of a Type of Generalised Hypergeometric Equation. By Professor T. M. MACROBERT, D.Sc. *Proc.*, vol. 59, pp. 49–54.

SECOND ORDINARY MEETING.

Monday, December 5, 1938.

Sir D'Arcy Wentworth Thompson, President, in the Chair.

The Minutes of the previous Ordinary Meeting were taken as read, and signed.

The DAVID ANDERSON-BERRY PRIZE (1938) and the Dr W. S. BRUCE MEMORIAL PRIZE (1938) were presented.

The PRESIDENT presented the DAVID ANDERSON-BERRY PRIZE to Dr MARY A. C. COWELL in the following terms:—

Our knowledge of the theoretical basis of the therapeutic application of X-rays is severely limited, and for this reason accurate data which throw light on this problem are of peculiar value.

Miss Cowell has conducted an extensive series of careful clinical observations based upon a highly accurate system of measurement of dosage. This work has provided new information as to the manner in which the effect produced is related to the duration and intensity of the exposure. The results indicate that there may be advantages in the treatment of tumours in the use of short exposures of relatively high intensity.

Such information has a great practical value in that it aids the search for the optimum method of employment of X-rays for therapeutic purposes, a subject in which the founder of the David Anderson-Berry Prize took such a keen personal interest.

The PRESIDENT then presented the Dr W. S. BRUCE MEMORIAL PRIZE to Mr ALEXANDER R. GLEN, the leader of the Oxford Expedition to Spitsbergen in 1933–34 and 1935–36.

This Prize is awarded by a Joint Committee of the Royal Physical Society, the Royal Scottish Geographical Society, and the Royal Society of Edinburgh.

Mr Glen was the leader of the Oxford University Arctic Expedition to New Friesland, Spitsbergen, in 1933 and 1934, and the Oxford Expedition to North East Land, Spitsbergen, in 1935 and 1936. The last-named expedition completed the survey of North East Land, explored the extent and thickness of the ice-sheet, examined the meteorological conditions, and investigated certain problems of the upper layers of the atmosphere.

He is the author of *Young Men in the Arctic* (1935); *Under the Pole Star* (1937); and *The Oxford University Arctic Expedition*, 1935, 1936 (published in the *Geographical Journal*, September and October 1937).

A billet of the Second Ordinary Meeting of the Royal Society of Edinburgh on Monday, December 16, 1878, was circulated by the President for the interest of the meeting. The fourth communication on this billet, of almost sixty years previously, was a "Note on Ulodendron and Halonia" by Mr D'ARCY WENTWORTH THOMPSON, communicated by Sir WYVILLE THOMSON. The President mentioned that the paper in question, when published (though not in the *Proceedings of the Royal Society of Edinburgh*), contained drawings executed by BENJAMIN PEACH, destined later to become the illustrious geologist.

The PRESIDENT then called upon Mr ALEXANDER R. GLEN, who delivered an address on "The Oxford University Arctic Expedition to North East Land, 1935–36."

Read by Title:—

1. The Structure and Function of the Alimentary Canal of some Tectibranch Molluscs, with a Note on Excretion. By VERA FRETTER, Ph.D. Communicated by Professor H. GRAHAM CANNON, F.R.S. *Trans.*, vol. 59, pp. 599–646.

2. Investigation of Visual Threshold Values. By Mrs E. M. BEATTIE. Communicated by Professor W. PEDDIE, D.Sc. *Proc.*, vol. 59, pp. 55–61.

THIRD ORDINARY MEETING.

Monday, January 9, 1939.

Sir D'Arcy Wentworth Thompson, President, in the Chair.

The Minutes of the previous Ordinary Meeting were taken as read, and signed.

At the request of the Council Mr F. I. G. RAWLINS, M.Sc., F.R.S.E., F.Inst.P. (Scientific Adviser to the Trustees, National Gallery, London), delivered an address on "Physical Methods in the Investigation of Paintings."

Read by Title:—

On the Invariance of Quantized Field Equations. By K. FUCHS, Ph.D. Communicated by Professor MAX BORN, M.A., Hon.D.Sc. *Proc.*, vol. 59, pp. 109–121.

FOURTH ORDINARY MEETING.

Monday, February 6, 1939.

Sir D'Arcy Wentworth Thompson, President, in the Chair.

The Minutes of the previous Ordinary Meeting were taken as read, and signed.

Professor E. T. WHITTAKER read a short address epitomising Professor P. A. M. DIRAC'S contributions to the mathematical theory of quantum mechanics.

The PRESIDENT presented to Professor P. A. M. DIRAC, F.R.S., the JAMES SCOTT PRIZE, and in the terms of the award and at the request of the Council called upon Professor DIRAC to deliver his Address entitled "The Relation between Mathematics and Physics." *Proc.*, vol. 59, pp. 122–129.

Professor MAX BORN and Professor WHITTAKER spoke at the conclusion of the Address.

Read by Title:—

Lunar Atmospheric Pressure Variations at Glasgow. By R. A. ROBB, M.A., D.Sc., and T. R. TANNAHILL, M.A., B.Sc. *Proc.*, vol. 59, pp. 81–90.

FIFTH ORDINARY MEETING.

Monday, March 6, 1939.

Sir D'Arcy Wentworth Thompson, President, in the Chair.

The Minutes of the previous Ordinary Meeting were taken as read, and signed.

The Ballot for the election of Ordinary Fellows then took place. The President nominated Dr A. C. STEPHEN and Dr MALCOLM WILSON as Scrutineers. The following gentlemen were declared duly elected:—

ARTHUR BARTON PILGRIM AMIES, WILLIAM GILLIES ANNAN, The Rev. JAMES HOUSTON BAXTER, CECIL ARNOLD BEEVERS, AMULYARATAN CHAKRAVARTI, JAMES DAVIDSON, FRANCIS DAVIES, VICTOR AMBROSE EYLES, IAN FRASER, JOHN GALLOWAY GALLOWAY, BIRENDRA NATH GHOSH, WILLIAM MICHAEL HERBERT GREAVES, WILLIAM ROBERT HALL, WALTER FEARN HARPER, NORMAN MILLER JOHNSON, STENARD ERNEST ANDREW LANDALE, CYRIL EDWARD LUCAS, JAMES WRIGHT MACFARLANE, JOHN IAN GRAHAM MACGREGOR, JAMES MACALISTER MACKINTOSH, JOHN WILLIAM MCNEE, WILLIAM MAIR, DONALD CAPELL MATHESON, JOHN BARRÉ DE WINTON MOLONY, ALASTAIR CAMPBELL MURRAY, WILLIAM DOUGLAS OLIPHANT, Sir ARTHUR OLVER, WILLIAM ANGUS SINCLAIR, BERNARD HALLEY STEWART, GEORGE MACFEAT WISHART, Sir HAROLD EDGAR YARROW, Bart.

Dr P. BACSICH, Dr G. M. WYBURN, and Mr A. W. M. BEVERIDGE signed the Roll and were admitted to Fellowship.

The PRESIDENT intimated that the MAKDOUGALL-BRISBANE PRIZE for the period 1934–1938 was awarded to Professor D. M. S. WATSON, F.R.S., for his paper entitled "On *Rhamphodopsis*, a Ptyctodont from the Middle Old Red Sandstone of Scotland" and for his many distinguished contributions to the science of Vertebrate Palæontology.

It was intimated that at the request of the Council and in terms of the BRUCE-PRELLER LECTURE FUND, Professor P. M. S. BLACKETT, F.R.S., University of Manchester, would deliver an Address on "The Mesotron: the New Unstable Cosmic Ray Particle," at the Ordinary Meeting of the Society on May 1, 1939.

The following Communications were submitted:—

1. Differential Fertility in Scotland. Part II. By ENID CHARLES, M.A., Ph.D. Communicated by Professor LANCELOT HOGBEN, F.R.S. *Trans.*, vol. 59, pp. 673–686.

2. The Fertility of Scottish Married Women, with special reference to the Period 1926–1935. By R. S. BARCLAY, B.Sc., and W. O. KERMACK, D.Sc., LL.D. *Proc.*, vol. 59, pp. 62–80.

3. Sources of Variation in Human Birth Weights. By H. P. DONALD, Ph.D. *Proc.*, vol. 59, pp. 91–108.

Read by Title:—

4. Relations between the Elliptic Cylinder Functions. By E. L. INCE, M.A., D.Sc. *Proc.*, vol. 59, pp. 176–183.

5. The Fresh-water Mollusca of the Tanganyika Territory and Zanzibar Protectorate. By ALAN MOZLEY, Ph.D. *Trans.*, vol. 59, pp. 687–744.

6. Studies on Reproduction in the Albino Mouse. II.—On the Maturation of Spermatozoa. By HUGO MERTON, Heidelberg. Communicated by Professor F. A. E. CREW, M.D., D.Sc. *Proc.*, vol. 59, pp. 207–218.

7. Cytogenetical Analysis of the Chromosomes in the Pig. By Professor F. A. E. CREW, M.D., D.Sc., and P. C. KOLLER, Ph.D., D.Sc. *Proc.*, vol. 59, pp. 163–175.

SIXTH ORDINARY MEETING.

Monday, May 6, 1939.

Sir D'Arcy Wentworth Thompson, President, in the Chair.

The Minutes of the previous Ordinary Meeting were taken as read, and signed.

The following signed the Roll and were admitted to Fellowship: Dr J. J. BLACKIE, Professor J. A. CARROLL, Dr C. A. BEEVERS, Mr JAMES DAVIDSON, Mr J. GALLOWAY GALLOWAY, Professor W. M. H. GREAVES, Dr S. E. A. LANDALE, Dr C. E. LUCAS, Dr J. W. MACFARLANE, Mr J. I. G. MACGREGOR, Mr WILLIAM MAIR, Professor D. C. MATHESON, Dr J. BARRÉ DE WINTON MOLONY, Sir ARTHUR OLVER, Mr W. A. SINCLAIR, and Dr BERNARD HALLEY STEWART.

It was intimated by the President that the following names had been proposed for Honorary Fellowship:

HARVEY (WILLIAMS) CUSHING, Emeritus Professor of Neurology, Yale School of Medicine, New Haven, Conn., U.S.A.

OTTO LOEWI, lately Professor of Pharmacology, University of Graz, Austria.

BERNARD LYOT, For. Assoc. Roy. Astron. Soc., l'Observatoire, Meudon (S. et O.), France.

At the request of the Council and in terms of the BRUCE-PRELLER LECTURE FUND, the PRESIDENT called upon Professor P. M. S. BLACKETT, F.R.S., University of Manchester, to deliver his Address on "The Mesotron: the New Unstable Cosmic Ray Particle."

Professor C. T. R. WILSON, Professor MAX BORN, and Professor E. T. WHITTAKER, contributed to the discussion, and the President moved a vote of thanks.

Read by Title:—

1. On Non-associative Combinations. By I. M. H. ETHERINGTON, B.A., Ph.D. *Proc.*, vol. 59, pp. 153–162.

2. Expressions for Generalised Hypergeometric Functions in Multiple Series. By Professor T. M. MACROBERT, D.Sc. *Proc.*, vol. 59, pp. 141–144.

3. The Molecular Spectra of the Hydrogen Isotopes. II.—The Assumption of a Common Potential Function for the Isotopic States. By IAN SANDEMAN, M.A., Ph.D. *Proc.*, vol. 59, pp. 130–140.

4. Further Contributions to our Knowledge of the Fossil Schizæaceæ (*Senftenbergia*). By NORMAN W. RADFORTH, M.A. Communicated by Professor J. WALTON, D.Sc. *Trans.*, vol. 59, pp. 745–761.

5. Tests of Significance of the Difference between Regression Coefficients derived from Two Sets of Correlated Variates. By F. YATES, Sc.D. Communicated by Dr A. C. AITKEN, F.R.S. *Proc.*, vol. 59, pp. 184–194.

SEVENTH ORDINARY MEETING.

Monday, June 5, 1939.

Sir D'Arcy Wentworth Thompson, President, in the Chair.

The Minutes of the previous Ordinary Meeting were taken as read, and signed.

Mr NORMAN MILLER JOHNSON signed the Roll and was admitted to Fellowship.

The following Communications were submitted:—

1. The Scottish Carboniferous Crinoidea. By JAMES WRIGHT, F.G.S. *Trans.*, vol. 60, pp. 1–78.

2. Some Geological Problems in Ardgour, Argyllshire. By H. I. DREVER, Ph.D. Communicated by Professor T. J. JEHU, M.D. *Trans.*, vol. 60, pp. 141–170.

Read by Title:—

3. The Arrangement of Fibre Follicles in some Mammals, with special reference to the Ovidæ. (Being selected and illustrated Notes made by the late Em. Professor J. E. Duerden.) Compiled by A. B. WILDMAN, B.Sc., Ph.D. Communicated by Professor F. A. E. CREW, F.R.S. *Trans.*, vol. 59, pp. 763–771.

4. An Early *Dictyoconus*, and the Genus *Orbitolina*: Their Contemporaneity, Structural Distinction, and Respective Natural Allies. By Lt.-Col. L. M. DAVIES, M.A., Ph.D., F.G.S. *Trans.*, vol. 59, pp. 773–790.

5. Studies on Reproduction in the Albino Mouse. III.—The Duration of Life of Spermatozoa in the Female Reproductive Tract. By Professor HUGO MERTON. Communicated by Professor F. A. E. CREW. *Proc.*, vol. 59, pp. 207–218.

EIGHTH AND LAST ORDINARY MEETING.

Monday, July 3, 1939.

Sir D'Arcy Wentworth Thompson, President, in the Chair.

The Minutes of the previous Ordinary Meeting were taken as read, and signed.

The following signed the Roll and were admitted to Fellowship: Dr MOWBRAY RITCHIE, Dr W. G. ANNAN, and Mr W. R. HALL.

The following were unanimously elected Honorary Fellows:—

HARVEY (WILLIAMS) CUSHING, Emeritus Professor of Neurology, Yale School of Medicine, New Haven, Conn., U.S.A.

OTTO LOEWI, lately Professor of Pharmacology, University of Graz, Austria.

BERNARD LYOT, For. Assoc. Roy. Astron. Soc., l'Observatoire, Meudon (S. et O.), France.

Mr W. WILLIAMSON and Mr A. GRAHAM DONALD acted as Scrutineers and were thanked by the President.

The PRESIDENT then called upon Dr MOWBRAY RITCHIE to deliver his Address given by request of the Council on "Radiation in Chemistry."

Read by Title:—

1. On the Reciprocation of Certain Matrices. By A. R. COLLAR, B.A., B.Sc. Communicated by Dr A. C. AITKEN, F.R.S. *Proc.*, vol. 59, pp. 195–206.

2. Some Ecological Aspects of the Intertidal Area of the Dee Estuary. By ALEC MILNE, M.A., B.Sc., Ph.D. Communicated by Professor LANCELOT HOGBEN, F.R.S. *Trans.*, vol. 60, pp. 107–139.

3. Genetic Algebras. By I. M. H. ETHERINGTON, B.A., Ph.D. *Proc.*, vol. 59, pp. 242–258.

4. Actiniaria and Zoantharia: Scottish National Antarctic Expedition, 1902–1904. By Professor OSKAR HENRIK CARLGREN. Communicated by Dr A. C. STEPHEN. *Trans.*, vol. 59, pp. 791–800.

5. Integration of a certain System of Linear Partial Differential Equations of Hypergeometric Type. By A. ERDÉLYI. Communicated by Professor E. T. WHITTAKER, F.R.S. *Proc.*, vol. 59, pp. 224–241.

PROCEEDINGS OF THE STATUTORY GENERAL MEETING

Ending the 156th Session, 1938–1939.

At the Statutory General Meeting, held in the Society's Rooms, 24 George Street, on Monday, October 23, 1939, at 4 P.M.

Sir D'Arcy Wentworth Thompson, President, in the Chair.

The Minutes of the Statutory Meeting held on October 24, 1938, were read, approved, and signed.

The President nominated as Scrutineers, for the election of Office-Bearers and Council, Professor F. W. SHARPLEY and Mr D. HALDANE.

The Bállot was then taken.

The General Secretary submitted the following report:—

GENERAL SECRETARY'S REPORT, OCTOBER 23, 1939.

During the session six Addresses have been given by request of the Council: on October 24, 1938, by Mr JOHN L. BAIRD on "The Development of Television"; on December 5, 1938, by Mr ALEXANDER R. GLEN on "The Oxford University Arctic Expedition to North East Land, 1935–36"; on January 9, 1939, by Mr F. I. G. RAWLINS on "Physical Methods in the Investigation of Painting"; on February 6, 1939, by Professor P. A. M. DIRAC, F.R.S., under the terms of the JAMES SCOTT PRIZE (1938), on "The Relation between Mathematics and Physics"; on May 1, 1939, by Professor P. M. S. BLACKETT, F.R.S., under the terms of the BRUCE-PRELLER LECTURE FUND (1939), on "The Mesotron: the New Unstable Cosmic Ray Particle"; and on July 3, 1939, by Dr MOWBRAY RITCHIE on "Radiation in Chemistry." Professor DIRAC's Lecture was published in *Proc. R.S.E.*, vol. 59, pp. 122–129.

33 papers were read as compared with 37 in the previous session. The papers were in the following subjects: Mathematics and Statistics, 13; Physics, 5; Chemistry, 1; Geology and Palæontology, 5; Zoology, 5; Animal Genetics, 4. 9 papers have been published in the *Transactions* and 22 in the *Proceedings*. 3 papers have been withdrawn, and several papers are still under the consideration of the Council.

The Society has lost by death 18 Fellows and 2 Honorary Fellows. 31 Fellows and 3 Foreign Honorary Fellows were elected. 1 Fellow resigned and 1 Fellow was removed from the Roll of Fellows.

Invitations were received and the Society was, or will be, represented as follows on the occasions mentioned:—

1. Réunion Internationale pour la Commemoration de la Découverte du Radium, Paris, November 26–30, 1938. Letter sent.
2. Maurice Caullery, Scientific Jubilee, Paris, February 25, 1939. Letter sent.
3. Lazzaro Spallanzani Celebration, University of Pavia, April 1939. Professor F. A. E. CREW, F.R.S.
4. Memorial Service for Sir ROBERT PHILIP, St Giles Cathedral, January 29, 1939. Lt.-Col. A. G. MCKENDRICK, Em. Professor R. STOCKMAN, and Lt.-Col. W. F. HARVEY.
5. Lord High Commissioner's Levee, Palace of Holyroodhouse, May 24, 1939. Professor F. A. E. CREW, Dr JAMES WATT, Dr A. C. AITKEN, and Dr. J. E. MACKENZIE.
6. Georgetown University, 150th Anniversary of Georgetown College, May 28 to June 5, 1939. Letter sent.
7. Centenaire de TH. RIBOT, Paris, June 1939. Letter sent.
8. National Association for the Prevention of Tuberculosis, June 29 to July 1, 1939, at Belfast. Letter sent.
9. Seventh International Congress of Genetics, Edinburgh, August 23–30, 1939. The PRESIDENT and Professor F. A. E. CREW.

10. Royal Swedish Academy of Sciences, 200th Anniversary, Stockholm, September 23–25, 1939. Address sent. Owing to the War our representative, Professor JAMES KENDALL, could not attend.
11. First Conference of the International Association for Scientific Tobacco Research, Bremen, September 25–30, 1939. Letter sent.
12. Royal Microscopical Society, London, Centenary Celebration, October 25–26, 1939. Mr WILLIAM WILLIAMSON. This Celebration has been postponed owing to the War.
13. Catholic University of America, 50th Anniversary, Washington, November 11–13, 1939. Professor HUGH S. TAYLOR, F.R.S.
14. Tenth All-India Oriental Conference, Hyderabad-Deccan, December 1939. Letter sent.
15. International Botanical Congress, Stockholm, July 17–25, 1940. Professor Sir W. WRIGHT SMITH, Professor J. WALTON, and Professor R. J. D. GRAHAM.
16. International Congress of Geology, London, July 31 to August 8, 1940. Dr G. W. TYRRELL and Dr J. E. RICHEY.

At the Ordinary Meeting on December 5, 1938, the Dr W. S. BRUCE MEMORIAL PRIZE (1938) was presented to Mr ALEXANDER R. GLEN; and the DAVID ANDERSON-BERRY PRIZE (1938) to Dr MARY A. C. COWELL.

Professor P. M. S. BLACKETT, F.R.S., was the BRUCE-PRELLER Lecturer, and Professor P. A. M. DIRAC, F.R.S., the JAMES SCOTT Lecturer during the session.

The MAKDOUGALL-BRISBANE PRIZE (1934–1938) was awarded to Professor D. M. S. WATSON, F.R.S., for his paper published in the *Transactions* of the Society, within the period, entitled "On *Rhamphodopsis*: a Ptyctodont from the Middle Old Red Sandstone of Scotland," and for his many distinguished contributions to Vertebrate Palæontology.

The JAMES GREGORY TERCENTENARY MEMORIAL VOLUME edited by Professor H. W. TURNBULL, F.R.S., has been published for the Society by Messrs G. Bell & Sons, Ltd., London. It includes his life and his mathematical work, which contains many advances, remarkable for the time and hitherto unrecorded, in the calculus and theory of numbers; also the important and vivid correspondence (1668–1675) between Gregory and his London friend Collins.

A grant of £200 has been generously awarded by the Carnegie Trust for the Universities of Scotland to Professor TURNBULL to guarantee against loss by the Society on publication of the volume.

The Subscribers to the President's Portrait Fund have presented to the PRESIDENT a smaller portrait of himself by Mr DAVID S. EWART, A.R.S.A. This portrait has been on exhibition in the Galleries of the Royal Scottish Academy during the summer.

Owing to the War and the possibility of damage, the Hume MSS. and two of the Society's pictures—John Robison, by Raeburn, and Sir Walter Scott, by Graham Gilbert—have been placed in safe custody outside our premises. The remainder of the pictures and certain other valuables have been placed in a strong room in the basement of our own building.

In response to a request from the Royal Society of London regarding the preparation of a National Service Central Register, the Council forwarded to the Royal Society a marked list of Fellows, so that the appropriate cards could be sent to them for completion.

Two large new book-cases were erected on the top floor to relieve congestion there.

The sum of £250 was expended on Library binding during the session, and arrears are gradually being overtaken.

A plaster cast of Napier of Merchiston has been presented to the Society by Mrs ROBERTSON, Kirkton, Dumfries, and is now placed in the Lecture Hall.

The cordial thanks of the Society are due to those who have made presentations to the Library.

Mr S. HEDDLE has retired from the post of Housekeeper to the Society, which he, with the late Mrs HEDDLE, so ably filled for a period of thirty years. An allowance has been granted to him. Mr Heddle has been succeeded by Mr WILLIAM BRYCE.

The acknowledgments of the Society are due to the Carnegie Trust for the Universities of Scotland for grants to authors towards the cost of illustrations, tabular matter, etc., of papers published in the *Transactions* and *Proceedings*, amounting to £187, 2s. 7d., to Birkbeck College, University of London, for a grant of £10 towards the cost of publication of Dr VERA FRETTER'S paper in *Transactions* (1939); to the University of St Andrews for £7 towards the cost of Professor PEACOCK and Miss ANN R. SANDERSON'S paper in the *Transactions* (1939); to Mr JAMES WRIGHT for £50 towards the cost of his paper in the *Transactions* (1939); to the London School of Hygiene and Tropical Medicine for £50 towards Dr ALAN MOZLEY'S paper in the *Transactions* (1939); and for £300 from the Government Publication Grant administered by the Royal Society of London, in aid of the cost of the Society's publications for the session 1938–1939.

TREASURER'S REPORT:—

The TREASURER, in submitting the Accounts for the past year, mentioned the leading items of Receipts and Expenditure, comparing them with those of the previous year, and stated that a bequest of £1000 made by Dr JAMES CURRIE, a former Treasurer, had been received during the session, the income of which would be used in the General Fund of the Society. The Treasurer also said that £460, 15s. 6d., including the £200 set aside in 1937–38, had been paid to the account of the GREGORY TERCENTENARY MEMORIAL VOLUME. The Treasurer indicated that the financial position of the Society was satisfactory.

Colonel J. C. LAMONT moved the adoption of the reports, and the reappointment of Messrs LINDSAY, JAMIESON & HALDANE, C.A., as auditors for the ensuing session. The meeting approved.

The Scrutineers reported that the Ballot Papers were in order, and the President declared that the following Office-Bearers and Members of Council had been duly elected:—

Professor E. T. WHITTAKER, M.A., Hon.Sc.D., D.Sc., LL.D., F.R.S., President.

Principal J. C. SMAIL, O.B.E., Companion Inst.E.E.	Vice-Presidents.
Professor J. WALTON, M.A., D.Sc.	
JAMES WATT, W.S., LL.D.	
LEONARD DOBBIN, Ph.D.	
JOHN ALEXANDER INGLIS, K.C., M.A., LL.B.	
Emeritus Professor RALPH STOCKMAN, M.D., LL.D., F.R.C.P.E.	

Professor JAMES P. KENDALL, M.A., D.Sc., F.R.S., General Secretary.

ALEXANDER C. AITKEN, M.A., D.Sc., F.R.S.	Secretaries to Ordinary Meetings.
Professor R. J. D. GRAHAM, M.A., D.Sc.	

E. MACLAGAN WEDDERBURN, O.B.E., LL.D., D.K.S., Treasurer.

JOHN E. MACKENZIE, D.Sc., Curator of Library and Museum.

MEMBERS OF COUNCIL.

Professor LANCELOT T. HOGBEN, M.A., D.Sc., F.R.S.

Professor JAMES RITCHIE, M.A., D.Sc.

G. W. TYRRELL, A.R.C.S., D.Sc., F.G.S.

Emeritus Professor C. T. R. WILSON, C.H., M.A., D.Sc., LL.D., F.R.S.

Professor R. C. GARRY, M.B., Ch.B., D.Sc.

Professor D. MURRAY LYON, M.D., F.R.C.P.E., D.P.H., D.Sc.

J. E. RICHEY, M.C., B.A., Sc.D., F.R.S., F.G.S.

The Hon. LORD ROBERTSON.

A. GRAHAM DONALD, M.A., F.F.A., F.S.A.Scot.

ALAN W. GREENWOOD, D.Sc., Ph.D.

Emeritus Professor T. H. MILROY, M.D., LL.D.

W. P. D. WIGHTMAN, Ph.D., M.Sc.

The Scrutineers were thanked for their services.

The retiring President welcomed the newly elected President of the Society, Professor E. T. WHITTAKER, F.R.S., and Professor Whittaker paid high tribute to the valuable services rendered to the Society by Sir D'ARCY WENTWORTH THOMPSON as President and in other offices.

Professor WHITTAKER also, in name of the Society, cordially thanked the other retiring Office-Bearers and Members of Council.

Sir D'ARCY WENTWORTH THOMPSON introduced Professor D. M. S. WATSON to the meeting, and presented to him the MAKDOUGALL-BRISBANE PRIZE for the period 1936–1938, for his paper published in the *Transactions* of the Society, within the period, entitled "On *Rhamphodopsis*, a Ptyctodont from the Middle Old Red Sandstone of Scotland," and for his many distinguished contributions to the science of Vertebrate Palæontology.

Professor WATSON was then called upon to deliver his Address on "The Origin of Frogs."

The CHAIRMAN cordially thanked Professor D. M. S. WATSON for his Address.

THE KEITH, MAKDOUGALL-BRISBANE, NEILL, GUNNING VICTORIA JUBILEE, JAMES SCOTT, BRUCE, AND DAVID ANDERSON-BERRY PRIZES, AND THE BRUCE-PRELLER LECTURE FUND.

The above Prizes will be awarded by the Council in the following manner:—

I. KEITH PRIZE.

The Keith Prize, consisting of a Gold Medal and about £30 in Money, will be awarded in the Session 1941–1942 for the "best communication on a scientific subject, communicated,* in the first instance, to the Royal Society of Edinburgh during the Sessions 1939–1940 and 1940–1941." Preference will be given to a Paper containing a discovery. (See also Council's resolutions at the end of these regulations.)

II. MAKDOUGALL-BRISBANE PRIZE.

(*Amended June* 7, 1926.)

This Prize is to be awarded biennially by the Council of the Royal Society of Edinburgh to such person, for such purposes, for such objects, and in such manner as shall appear to them the most conducive to the promotion of the interests of science; with the *proviso* that the Council shall not be compelled to award the Prize unless there shall be some individual engaged in scientific pursuit, or some Paper written on a scientific subject, or some discovery in science made during the biennial period, of sufficient merit or importance in the opinion of the Council to be entitled to the Prize.

1. The Prize, consisting of a Gold Medal and a sum of Money, will be awarded before the close of the Session 1940–1941, for an Essay, Paper, or other work having reference to any branch of scientific inquiry, either material or mental.
2. It is open to all men of science.
3. The specific subjects taken into consideration in the current award are governed by the resolutions of the Council as stated at the end of these regulations.
4. For the current period the Committee is representative of Group *A*.
5. The Committee will consider Papers presented to the Society within the Sessions 1938–1939 and 1939–1940, and will make a recommendation.

It is empowered to recommend either:—

(*a*) An award to the Author of an Essay or Paper considered as above, or
(*b*) That no award be made on the ground that, within its group, no Paper of sufficient merit has been presented, or

* For the purposes of this award the word "communicated" shall be understood to mean the date on which the manuscript of a Paper is received in its final form for printing, as recorded by the General Secretary or other responsible official.

(*c*) That the Prize be awarded to some distinguished man of learning, who may not have presented a Paper to the Society within the period considered, but who is willing to deliver an address.

III. NEILL PRIZE.

The Council of the Royal Society of Edinburgh having received the bequest of the late Dr PATRICK NEILL of the sum of £500, for the purpose of "the interest thereof being applied in furnishing a Medal or other reward every second or third year to any distinguished Scottish Naturalist, according as such Medal or reward shall be voted by the Council of the said Society," hereby intimate:—

1. The NEILL PRIZE, consisting of a Gold Medal, will be awarded during the Session 1941–1942.

2. The Prize will be given for a Paper of distinguished merit, on a subject of Natural History, by a Scottish Naturalist, which shall have been presented * to the Society during the two years preceding the fourth Monday in October 1941,—or failing presentation of a Paper sufficiently meritorious, it will be awarded for a work or publication by some distinguished Scottish Naturalist, on some branch of Natural History, bearing date within five years of the time of award. (See also resolutions of Council at the end of these regulations.)

IV. GUNNING VICTORIA JUBILEE PRIZE.

This Prize, founded in the year 1887 by Dr R. H. GUNNING, is to be awarded quadrennially by the Council of the Royal Society of Edinburgh, in recognition of original work in Physics, Chemistry, or Pure or Applied Mathematics.

Evidence of such work may be afforded either by a Paper presented * to the Society, or by a Paper on one of the above subjects, or some discovery in them elsewhere communicated or made, which the Council may consider to be deserving of the Prize.

The Prize consists of a sum of money, and is open to men of science resident in or connected with Scotland. The first award was made in the year 1887. The next award will be made in Session 1940–1941.

In accordance with the wish of the Donor, the Council of the Society may on fit occasions award the Prize for work of a definite kind to be undertaken during the three succeeding years by a scientific man of recognised ability.

V. JAMES SCOTT PRIZE.

This Prize, founded in the year 1918 by the Trustees of the JAMES SCOTT Bequest, is to be awarded triennially, or at such intervals as the Council of the Royal Society of Edinburgh may decide, "for a lecture or essay on the fundamental concepts of Natural Philosophy."

* For the purposes of this award the word "presented" shall be understood to mean the date on which the manuscript of a Paper is received in its final form for printing, as recorded by the General Secretary or other responsible official.

VI. BRUCE PRIZE.

The Society is trustee of a fund, instituted in 1923, to commemorate the work of Dr W. S. BRUCE as an explorer and scientific investigator in polar regions.

The Committee of Award is appointed jointly by the Royal Society of Edinburgh, the Royal Physical Society, and the Royal Scottish Geographical Society.

The Prize consists of a Bronze Medal and sum of Money. It is open to workers of all nationalities, with a preference, *ceteris paribus*, for those of Scottish birth or origin, and is to be awarded biennially for some notable contribution to Natural Sciences, such as Zoology, Botany, Geology, Meteorology, Oceanography, and Geography; the contribution to be in the nature of new knowledge, the outcome of a personal visit to polar regions on the part of the recipient. The recipient shall preferably be at the outset of his career as an investigator.

The next award will be made in 1942. Papers for the consideration of the Committee should be in the hands of the General Secretary of the Royal Society, 22 George Street, Edinburgh 2, not later than March 31 of that year.

VII. BRUCE-PRELLER LECTURE FUND.

The Council of the Royal Society of Edinburgh having received in 1929 the bequest of the late Dr CHARLES DU RICHE PRELLER of the sum of £500, decided that the income thereof be applied by the Council biennially as an honorarium for a special BRUCE-PRELLER Lecture or Address by an outstanding man of science, its subject to be Geology or Electrical or Physical Science, or in the discretion of the Council some other branch of science. The next award will be made in Session 1940–1941.

VIII. DAVID ANDERSON-BERRY FUND.

The Council of the Royal Society of Edinburgh having received in the year 1930, free of duty, the capital sum of one thousand pounds (£1000), to be used in terms of the will of the late Dr DAVID ANDERSON-BERRY, dated 23rd April 1926, decided that the income thereof be applied triennially, "in the first place in the presentation of a gold medal, and in the second place in the payment of a sum of money to the winner for the year of such gold medal, the winner being the person who, in the opinion of the Society, shall be the producer for the year of the best essay on the nature of X-rays and their therapeutical effect on human diseases."

The third award will be made in July 1941.

RESOLUTIONS OF COUNCIL IN REGARD TO THE MODE OF AWARDING PRIZES.

(*See Minutes of Meeting of January* 18, 1915.)

I. With regard to the Keith and Makdougall-Brisbane Prizes, which are open to all Sciences, the mode of award will be as follows:—

1. Papers or essays to be considered shall be arranged in two groups, *A* and *B*—Group *A* to include Astronomy, Chemistry, Mathematics, Metallurgy, Meteorology, and Physics; Group *B* to include Anatomy, Anthropology, Botany, Geology, Pathology, Physiology, and Zoology.
2. These two prizes shall be awarded to each group in alternate biennial periods, provided Papers worthy of recommendation have been communicated to the Society.
3. Prior to the adjudication the Council shall appoint, in the first instance, a Committee composed of representatives of the group of Sciences which did not receive the award in the immediately preceding period. The Committee shall consider the Papers which come within their group of Sciences, and report in due course to the Council.
4. In the event of the aforesaid Committee reporting that within their group of subjects there is, in their opinion, no Paper worthy of being recommended for the award, the Council, on accepting this report, shall appoint a Committee representative of the alternate group to consider Papers coming within their group and to report accordingly.
5. Papers to be considered by the Committees shall fall within the period dating from the last award in Groups *A* and *B* respectively.

II. With regard to the Neill Prize, the term "Naturalist" shall be understood to include any student in the Sciences composing Group *B*, namely, Anatomy, Anthropology, Botany, Geology, Pathology, Physiology, Zoology.

AWARDS OF THE KEITH, MAKDOUGALL-BRISBANE, NEILL, GUNNING, JAMES SCOTT, BRUCE, AND DAVID ANDERSON-BERRY PRIZES, AND THE BRUCE-PRELLER LECTURE FUND.

(*Earlier Awards will be found printed in the preceding volumes of the Proceedings.*)

I. KEITH PRIZE.

49TH BIENNIAL PERIOD, 1923–25.—HERBERT WESTREN TURNBULL, M.A., for the papers on "Hyper-Algebra," "Invariant Theory," and "Algebraic Geometry," three of which have been published in the Proceedings within the period of award.

50TH BIENNIAL PERIOD, 1925–27.—THOMAS JOHN JEHU, M.A., M.D., F.G.S., and ROBERT MELDRUM CRAIG, M.A., B.Sc., F.G.S., for the joint series of papers which have recently appeared in the Transactions of the Society on the "Geology of the Outer Hebrides."

51ST BIENNIAL PERIOD, 1927–29.—CHRISTINA C. MILLER, B.Sc., Ph.D., for her papers on the "Slow Oxidation of Phosphorus Trioxide," published in the Proceedings within the period of the award, and in consideration of subsequent developments on "Slow Oxidation of Phosphorus," published elsewhere.

52ND BIENNIAL PERIOD, 1929–31.—ALAN WILLIAM GREENWOOD, M.Sc., Ph.D., for his papers on the "Biology of the Fowl," several of which have appeared in the Proceedings within the period of award.

53RD BIENNIAL PERIOD, 1931–33.—A. CRICHTON MITCHELL, D.Sc., for his paper "On the Diurnal Incidence of Disturbance in the Terrestrial Magnetic Field," published in the Transactions within the period of award.

54TH BIENNIAL PERIOD, 1933–35.—LANCELOT T. HOGBEN, D.Sc., F.R.S., for his papers on genetical subjects, published alone and in collaboration, which have appeared in the Proceedings within the period of award.

55TH BIENNIAL PERIOD, 1935–37.—HAROLD STANLEY RUSE, M.A., for his paper "On the Geometry of Dirac's Equations and their Expression in Tensor Forms," published in the Proceedings within the period, and for his other papers in the Proceedings.

II. MAKDOUGALL-BRISBANE PRIZE.

33RD BIENNIAL PERIOD, 1922–24.—Professor H. STANLEY ALLEN, D.Sc., for his papers on the "Quantum and Atomic Theory," published in the Society's Proceedings within the period.

34TH BIENNIAL PERIOD, 1924–26.—CHARLES MORLEY WENYON, C.M.G., C.B.E., F.R.S., for his distinguished work in Protozoology extending over a period of twenty-one years.

35TH BIENNIAL PERIOD, 1926–28.—W. O. KERMACK, M.A., D.Sc., for his contributions to Chemistry, published in the Society's Proceedings and elsewhere.

36TH BIENNIAL PERIOD, 1928–30.—NELLIE B. EALES, D.Sc., for her papers in the Society's Transactions on "The Anatomy of a Fœtal African Elephant."

37TH BIENNIAL PERIOD, 1930–32.—A. C. AITKEN, M.A., D.Sc., for various contributions to Mathematics, published in the Society's Proceedings and elsewhere.

38TH BIENNIAL PERIOD, 1932–34.—A. E. CAMERON, M.A., D.Sc., for his publications in Entomology, including his paper in the Transactions, "The Life-History and Structure of *Hæmatopota pluvialis* Linné (Tabanidæ)."

39TH BIENNIAL PERIOD, 1934–36.—E. M. ANDERSON, M.A., D.Sc., for his paper "The Dynamics of the Formation of Cone-sheets, Ring-dykes, and Caldron-subsidences," published in the Society's Proceedings within the period.

40TH BIENNIAL PERIOD, 1936–38.—Professor D. M. S. WATSON, F.R.S., for his paper published in the Society's Transactions on "*Rhamphodopsis*, a Ptyctodont from the Middle Old Red Sandstone of Scotland" and for his distinguished contributions to Vertebrate Palæontology.

III. THE NEILL PRIZE.

9TH BIENNIAL PERIOD, 1923–25.—FREDERICK ORPEN BOWER, F.R.S., for his recent contributions to Botanical knowledge and in recognition of his published work extending over a period of forty-five years.

10TH BIENNIAL PERIOD, 1925–27.—ARTHUR ROBINSON, M.D., M.R.C.S., for his contributions to Comparative Anatomy and Embryology.

11TH BIENNIAL PERIOD, 1927–29.—EDWARD BATTERSBY BAILEY, M.C., F.R.S., in recognition of his valuable contributions to the Geology of Scotland, two of which have recently appeared in the Transactions of the Society.

12TH BIENNIAL PERIOD, 1929–31.—CHARLES HENRY O'DONOGHUE, D.Sc., for his papers on the "Blood Vascular System," and for his earlier work on the "Morphology of the *corpus luteum*."

13TH BIENNIAL PERIOD. 1931–33.—GEORGE WALTER TYRRELL, A.R.C.S., D.Sc., for his contributions to the Geology and Petrology of Sub-Arctic and Sub-Antarctic Lands.

14TH BIENNIAL PERIOD, 1933–35.—SAMUEL WILLIAMS, Ph.D., for his contributions to the Anatomy and Experimental Morphology of the Pteridophyta.

15TH BIENNIAL PERIOD, 1935–37.—WILLIAM J. HAMILTON, M.D., D.Sc., for his contributions to the Embryology of the Ferret and other work published in the Transactions.

IV. GUNNING VICTORIA JUBILEE PRIZE.

8TH QUADRENNIAL PERIOD, 1924–28.—Professor E. T. WHITTAKER, F.R.S., in recognition of his distinguished contributions to Mathematical Science, and of his promotion of Mathematical Research in Scotland.

9TH QUADRENNIAL PERIOD, 1928–32.—Emeritus Professor Sir J. WALKER, F.R.S., for numerous contributions to Physical and General Chemistry.

10TH QUADRENNIAL PERIOD, 1932–36.—Professor C. G. DARWIN, F.R.S., for his distinguished contributions in Mathematical Physics.

V. JAMES SCOTT PRIZE.

1ST AWARD, 1918–22.—Professor A. N. WHITEHEAD, F.R.S., for his lecture delivered on June 5, 1922, on "The Relatedness of Nature."

2ND AWARD, 1922–27.—Sir JOSEPH LARMOR, M.A., D.Sc., LL.D., F.R.S., for his lecture delivered on July 4, 1927, on "The Grasp of Mind on Nature."

3RD AWARD, 1927–30.—Professor NIELS BOHR, for his lecture delivered on May 26, 1930, on "Philosophical Aspects of Atomic Theory."

4TH AWARD, 1930–33.—Professor Dr ARNOLD SOMMERFELD, for his lecture delivered on May 1, 1933, on "Ways to the Knowledge of Nature."

5TH AWARD, 1933–38.—Professor P. A. M. DIRAC, F.R.S., for his lecture delivered on February 6, 1939, on "The Relation between Mathematics and Physics."

VI. BRUCE PRIZE.

1ST AWARD, 1926.—JAMES MANN WORDIE, M.A., for his Oceanographical and Geological work in both Polar Regions.

2ND AWARD, 1928.—H. U. SVERDRUP, for his contributions to the knowledge of the Meteorology, Magnetism, and Tides of the Arctic, as an outcome of his travels with the Norwegian Expedition in the "Maud" from 1918 to 1925.

3RD AWARD, 1930.—N. A. MACKINTOSH, M.Sc., A.R.C.S., for his researches into the Biology of Whales in the Waters of the Falkland Islands Dependencies.

4TH AWARD, 1932.—HENRY GINO WATKINS, for important contributions to the topography of Spitsbergen, Labrador, and East Greenland, and investigation of the Ice Cap of Greenland.

5TH AWARD, 1936.—JAMES WILLIAM SLESSER MARR, M.A., B.Sc., for his work in the Southern Ocean and more particularly for his monograph on the South Orkney Islands.

6TH AWARD, 1938.—ALEXANDER R. GLEN, for his work in Spitsbergen, including Survey in New Friesland and the completion of the map of North East Land.

VII. BRUCE-PRELLER LECTURE FUND.

1ST AWARD, 1931.—Professor E. T. WHITTAKER, F.R.S., for his lecture, "James Clerk Maxwell and Mechanical Descriptions of the Universe."

2ND AWARD, 1933.—Professor C. H. LANDER, C.B.E., for his lecture on October 23, 1933, on "The Liquefaction of Coal."

3RD AWARD, 1935.—Professor W. L. BRAGG, O.B.E., F.R.S., for his lecture on February 4, 1935, on "The New Crystallography."

4TH AWARD, 1937.—Professor H. S. TAYLOR, F.R.S., for his lecture on June 7, 1937, on "Heavy Hydrogen in Scientific Research."

5TH AWARD, 1939.—Professor P. M. S. BLACKETT, F.R.S., for his lecture on May 1, 1939, on "The Mesotron: the New Unstable Cosmic Ray Particle."

VIII. DAVID ANDERSON-BERRY FUND.

1ST AWARD, 1935.—CHARLES MELVILLE SCOTT, M.A., M.B., D.Sc., for his essay "On the Action of X- and Gamma-Rays on Living Cells."

2ND AWARD, 1938.—MARY A. C. COWELL, M.B., Holt Radium Institute, Manchester, for her essay "An Investigation into some of the Factors affecting the Response of Human Skin and Human Skin Tumours to Radiation."

ABSTRACT

OF

THE ACCOUNTS

OF

THE ROYAL SOCIETY OF EDINBURGH,

SESSION—1st October 1938 to 30th September 1939.

E. MACLAGAN WEDDERBURN, O.B.E., D.K.S., LL.D.
Treasurer.

I. GENERAL FUND

1. *Funds as at 30th September* 1938, per last Account				£13,357 16 1
Less—Reserve for Gregory Memorial Volume			£200 0 0	
Contributions in advance for 1938–39			9 9 0	
				209 9 0
				£13,148 7 1
2. *James Currie Bequest*				1,000 0 0
3. *Commutation Fee Fund*—				
1 Fellow elected Session 1931–32				44 2 0
				£14,192 9 1
Add **Income**—				
1. *Contributions for current Session*—				
503 Fellows at £3, 3s. each		£1584 9 0		
Fees of Admission of thirty-one New Fellows		97 13 0		
First Annual Contribution of thirty-one New Fellows		97 13 0		
			£1779 15 0	
2. *Extra Contributions for* 1938–39 under Amended Law VI.—				
Voluntary Contributions		£36 15 0		
Commutation		10 10 0		
			47 5 0	
3. *Interest Received, Untaxed*—				
General Fund—				
On £2100 2½% Consolidated Stock		£52 10 0		
On £9742, 12s. 5d. 2¾% Guaranteed Stock (1933)		267 18 4		
On Deposit Receipts		10 10 10		
		£330 19 2		
Special Subscription Fund—				
On £936, 0s. 11d. 2¾% Guaranteed Stock (1933)		25 14 8		
R. M. Smith Legacy—				
On £554, 13s. 7d. 2¾% Guaranteed Stock (1933)		15 5 0		
Publication Fund—				
On £2568, 5s. 1d. 2¾% Guaranteed Stock (1933)		70 12 4		
Commutation Fee Fund—				
On £198, 8s. 11d. 2¾% Guaranteed Stock (1933)	5 9 0			
On Deposit Receipts	1 2 9			
		6 11 9		
			449 2 11	
Forward			£2276 2 11	£14,192 9 1

	Forward	£2276 2 11	£14,192 9 1
4. *Transactions and Proceedings sold*		133 11 11	
5. *Grants*—			
Annual Grant from Government . .	£600 0 0		
Grant from Royal Society from Government Publication Grant	300 0 0		
Grants from Trustees of the Carnegie Trust for the Universities of Scotland	187 2 7		
Other Sources	117 0 0		
		1204 2 7	
6. *Publication Fund*—Sale of Volumes . . .		2 8 10	
			3,616 6 3
			£17,808 15 4
Deduct **Expenditure**—			
1. *Expenses of Transactions and Proceedings*—			
Transactions		£662 2 4	
Proceedings		590 3 3	
		£1252 5 7	
2. *Books, Periodicals, Newspapers, etc.*		247 18 3	
3. *Library Binding*		251 12 1	
4. *Gregory Memorial Volume*—			
Sums paid to Account	£460 15 6		
Less—Reserve made 1937–38 . .	200 0 0		
		260 15 6	
5. *General Upkeep of Society's Rooms*—			
Insurance and Water Rates . . .	£28 5 2		
Repairs and Furnishings	40 2 2		
Heating and Lighting.	60 14 0		
Cleaning	14 5 3		
Caretaker's Salary and Uniform . .	149 3 4		
Charwomen's Wages	82 16 8		
		375 6 7	
6. *Management*—			
Honorarium to General Secretary. .	£100 0 0		
Salaries	565 0 0		
Fee to Treasurer's Clerk	35 0 0		
Society's Contribution to Staff Pension Fund under Universities Scheme .	57 10 0		
State Insurance	4 2 4		
General Printing and Stationery . .	134 12 1		
Telephone	19 18 6		
Audit Fee	10 10 0		
Tea Expenses at Meetings . . .	37 18 6		
Postages and Petty Outlays . . .	56 17 9		
Miscellaneous	19 3 3		
S. Heddle allowance	33 6 8		
		1073 19 1	
7. *Extraordinary Expenditure*—			
Repairs to Caretaker's House—painting and plumbing	£52 0 4		
Additional Shelving	46 10 0		
		98 10 4	
8. *James Currie Bequest*—			
On Deposit Receipt	£1000 0 0		
9. *Commutation Fee Fund*—			
Transferred to Special Fund (p. 311) .	£44 2 0		
10. *Arrears of Contributions* written off as irrecoverable .		12 12 0	
			3,572 19 5
		Funds as at 30th September 1939	£14,235 15 11

Represented by—

£9742, 12s. 5d. 2½% Guaranteed Stock (1933) at cost		£8,298 1 9	
£2100 2½% Consolidated Stock at cost		1,113 0 0	
£554, 13s. 7d. 2½% Guaranteed Stock (1933) at cost—Robert Mackay Smith Legacy		472 8 9	
£2568, 5s. 1d. 2½% Guaranteed Stock (1933) at cost—Publication Fund (comprising Peter Guthrie Tait Memorial Fund and Dr John Aitken Fund) .		2,187 9 1	
£936, 0s. 11d. 2½% Guaranteed Stock (1933) at cost—Special Subscription Fund		797 5 2	
Due by Union Bank of Scotland, Ltd., on Deposit Receipt—James Currie Bequest		1000 0 0	
£198, 8s. 11d. 2½% Guaranteed Stock (1933) at cost	£154 7 0		
Due by Union Bank of Scotland, Ltd., on Deposit Receipt . . .	89 5 0		
Life Membership Commutation Fee Fund . .		243 12 0	
Arrears of Contribution at 30th September 1939—			
Present Session	£100 16 0		
Previous Session	31 10 0		
		132 6 0	
Cash—Imprest Amount		15 0 0	
		£14,259 2 9	
Less—Due to Union Bank of Scotland, Ltd., on Current Account .	£17 0 10		
Contributions in Advance for 1939–40	6 6 0		
		23 6 10	
			£14,235 15 11

In addition the Society owns the Library, Museum, Pictures, etc., and Furniture in the Rooms 22 George Street, Edinburgh.

Note—

Income for current Session as per preceding Accounts		£3616 6 3
Add—Arrears of Contributions at 30th September 1938 . . .		94 10 0
		£3710 16 3
Deduct—*Expenditure* for current Session as per preceding Accounts	£3572 19 5	
Arrears of Contributions at 30th September 1939	132 6 0	
		3705 5 5
Surplus for year to 30th September 1939		£5 10 10

COMMUTATION FEE FUND.

Year to 30th September 1939.

Funds as at 30th September 1938, per last Account		£199 10 0
Fee received for Life Membership—		
1 Fellow elected 1931–32		44 2 0
Interest received—On Investment	£5 9 0	
On Deposit Receipt	1 2 9	
		6 11 9
		£250 3 9
Deduct—		
Transferred to General Fund (Interest received)		6 11 9
Total Funds . . . (included in General Funds, above)		£243 12 0

Represented by—

£198, 8s. 11d. 2½% Guaranteed Stock (1933) at cost .	£154 7 0	
Due by Union Bank of Scotland, Ltd., on Deposit Receipt .	89 5 0	
		243 12 0

PUBLICATION FUND.

(*Comprising Peter Guthrie Tait Memorial Fund and Dr John Aitken Fund.*)

Year to 30th September 1939.

Funds as at 30th September 1938, per last Account		£2187 9 1
Interest Received: Untaxed—		
Peter Guthrie Tait Memorial Fund—		
On £1929, 17s. 1d. 2½% Guaranteed Stock (1933)	£53 1 4	
Dr John Aitken Fund—		
On £638, 8s. 2½% Guaranteed Stock (1933)	17 11 0	
		70 12 4
Sale of Volumes, Dr John Aitken Fund		2 8 10
		£2260 10 3
Deduct—		
Transferred to General Fund to meet cost of Publication—		
Interest, p. 309	£70 12 4	
Sale of Volumes, p. 310	2 8 10	
		73 1 2
Total Funds (included in General Fund, p. 311)		£2187 9 1
Represented by—		
£2568, 5s. 1d. 2½% Guaranteed Stock (1933) at cost		£2187 9 1

II. PRIZE FUNDS

Year to 30th September 1939.

1. Keith Fund—			
Funds as at 30th September 1938, per last Account—			
£801, 13s. 4d. 2½% Guaranteed Stock (1933) at cost			£682 16 1
Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt			38 3 2
			£720 19 3
Interest Received—On Investment		£22 0 10	
On Deposit Receipts		0 10 2	
			22 11 0
			£743 10 3
2. Neill Fund—			
Funds as at 30th September 1938, per last Account—			
£370 2½% Guaranteed Stock (1933) at cost		£315 2 9	
Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt		18 2 4	
		£333 5 1	
Interest received—On Investment	£10 3 6		
On Deposit Receipts	0 4 7		
		10 8 1	
			343 13 2
3. Makdougall-Brisbane Fund—			
Funds as at 30th September 1938, per last Account—			
£493, 6s. 7d. 2½% Guaranteed Stock (1933) at cost		£420 3 8	
Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt		52 11 11	
		472 15 7	
Interest received—On Investment	£13 11 2		
On Deposit Receipts	0 12 0		
		14 3 2	
			486 18 9
		Forward	£1574 2 2

		Forward	£1574 2 2
4. Makerstoun Magnetic Meteorological Observation Fund—			
Funds as at 30th September 1938, per last Account—			
£308, 6s. 9d. 2½% Guaranteed Stock (1933) at cost .		£262 12 5	
Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt		156 17 2	
		£419 9 7	
Interest received—On Investment . .	£8 9 6		
On Deposit Receipts .	1 14 2		
		10 3 8	
			429 13 3
5. Gunning Victoria Jubilee Prize Fund (Instituted by Dr Gunning of Edinburgh and Rio de Janeiro)—			
Funds as at 30th September 1938, per last Account—			
£739, 12s. 5d. 2½% Guaranteed Stock (1933) at cost .		£629 19 2	
Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt		100 0 9	
		£729 19 11	
Interest received—On Investment . .	£20 6 8		
On Deposit Receipts .	1 3 4		
		21 10 0	
			751 9 11
6. James Scott Prize Fund—			
Funds as at 30th September 1938, per last Account—			
£305, 5s. 2½% Guaranteed Stock (1933) at cost . .		£259 19 10	
Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt		45 13 4	
		£305 13 2	
Interest received—On Investment . .	£8 7 10		
On Deposit Receipts .	0 4 1		
		8 11 11	
		£314 5 1	
Deduct—Professor P. A. M. Dirac 1933-38 Award .		£40 0 0	
			274 5 1
7. Dr W. S. Bruce Memorial Fund—			
Funds as at 30th September 1938, per last Account—			
£289, 1s. 5d. 2½% Guaranteed Stock (1933) at cost .		£246 4 2	
Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt		55 11 5	
		£301 15 7	
Interest received—On Investment . .	£7 18 10		
On Deposit Receipts .	0 9 10		
		8 8 8	
		£310 4 3	
Deduct—Alexander R. Glen 1938 Award .	£10 0 0		
Cost of Bronze Medal . . .	0 15 0		
		10 15 0	
			299 9 3
8. Bruce-Preller Lecture Fund—			
Funds as at 30th September 1938, per last Account—			
£140, 9s. Royal Bank of Scotland Stock, taken over at 350%		£491 11 6	
Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt		57 9 11	
	Forward	£549 1 5	£3328 19 8

Forward		£549 1 5	£3328 19 8
Interest received—On Investment (less tax £6, 11s. 4d.)	£17 6 2		
On Deposit Receipts	0 10 4		
		17 16 6	
Repayment of Income Tax for year to December 1938		6 11 4	
		£573 9 3	
Deduct—Professor P. M. S. Blackett 1936–38 Award		40 0 0	
			533 9 3
9. Dr David Anderson-Berry Fund—			
Funds as at 30th September 1938, per last Account—			
£1528, 0s. 4d. Local Loans 3% Stock at cost		£1000 0 0	
Balance due by Union Bank of Scotland, Ltd., on Deposit Receipt		128 8 9	
		£1128 8 9	
Interest received—On Investment	£45 16 8		
On Deposit Receipts	0 10 9		
		46 7 5	
		£1174 16 2	
Deduct—Dr Mary A. C. Cowell 1938 Award	£100 0 0		
Cost of Gold Medal	22 4 8		
		122 4 8	
			1052 11 6
Total Funds			£4915 0 5
Represented by—			
£3307, 5s. 6d. 2½% Guaranteed Stock (1933) at cost		£2816 18 1	
£1528, 0s. 4d. Local Loans 3% Stock at cost		1000 0 0	
£140, 9s. Royal Bank of Scotland Stock, taken over at 350%		491 11 6	
Due by Union Bank of Scotland, Ltd., on Deposit Receipt		606 10 10	
			£4915 0 5

Note.—Under the Will of the late Professor Charles Piazzi Smyth and his wife, the Society will, on the expiry of certain liferents, become entitled to payment of the residue to be applied as set out in the will.

EDINBURGH, 13*th October* 1939.—We have examined the preceding Accounts of the Treasurer of the Royal Society of Edinburgh for the Session 1938–1939, and have found them to be correct. The Investments held for the General Fund and the Prize Funds as shown in the foregoing Accounts have been verified by us as at 30th September 1939.

LINDSAY, JAMIESON & HALDANE, C.A.
Auditors.

VOLUNTARY CONTRIBUTOR who has made a Single Payment under Law VI (end of para. 3).

Dr A. C. Cumming £10 10 0

VOLUNTARY CONTRIBUTORS under Law VI (end of para. 3), to 30th September 1939.

Professor F. O. Bower, F.R.S.	£1	1	0
Dr G. S. Brock	1	1	0
Col. D. Carnegie	1	1	0
Professor E. G. Coker, F.R.S.	1	1	0
Professor J. N. Collie, F.R.S.	1	1	0
W. B. Coutts, Esq.	1	1	0
D. B. Dott, Esq.	1	1	0
L. M. Douglas, Esq.	1	1	0
Dr E. O. Fergus	1	1	0
Dr R. A. Fleming	1	1	0
Alex. Fraser, Esq.	1	1	0
Wm. Fraser, Esq.	1	1	0
Sir Alexander Gibb, F.R.S.	1	1	0
Sir J. Graham Kerr, F.R.S.	1	1	0
Dr W. F. Hume	1	1	0
Dr G. R. Jeffrey	1	1	0
F. H. Lightbody, Esq.	1	1	0
Dr G. McGowan	1	1	0
	£18	18	0

Carried forward	£18	18	0
Dr Arch. McKendrick	1	1	0
Dr D. J. Mackintosh	1	1	0
Professor J. MacKinnon	1	1	0
Dr H. R. Mill	1	1	0
Professor J. Miller	1	1	0
Dr A. Morgan	1	1	0
R. C. Mossman, Esq.	1	1	0
A. G. Ramage, Esq.	1	1	0
E. Smart, Esq.	1	1	0
J. R. Ratcliffe, Esq.	1	1	0
C. A. Stevenson, Esq.	1	1	0
Dr H. F. Stockdale	1	1	0
G. S. Thomson, Esq.	1	1	0
R. T. Thomson, Esq.	1	1	0
Very Rev. L. MacLean Watt	1	1	0
W. Williamson, Esq.	1	1	0
A. Wilson, Esq.	1	1	0
	£36	15	0

Single Payment	£10	10	0
Other Payments	36	15	0
Total	£47	5	0

THE COUNCIL OF THE SOCIETY.

23rd October 1939.

PRESIDENT.

PROFESSOR E. T. WHITTAKER, M.A., Hon.Sc.D., D.Sc., LL.D., F.R.S.

VICE-PRESIDENTS.

PRINCIPAL J. C. SMAIL, O.B.E., Companion Inst.E.E.
PROFESSOR JOHN WALTON, M.A., D.Sc.
JAMES WATT, W.S., LL.D.
LEONARD DOBBIN, Ph.D.
JOHN ALEXANDER INGLIS, K.C., M.A., LL.B.
EMERITUS PROFESSOR R. STOCKMAN, M.D., LL.D., F.R.C.P.E.

GENERAL SECRETARY.

PROFESSOR JAMES P. KENDALL, M.A., D.Sc., F.R.S.

SECRETARIES TO ORDINARY MEETINGS.

ALEXANDER C. AITKEN, M.A., D.Sc., F.R.S.
PROFESSOR R. J. D. GRAHAM, M.A., D.Sc.

TREASURER.

E. MACLAGAN WEDDERBURN, O.B.E., LL.D., D.K.S.

CURATOR OF LIBRARY AND MUSEUM.

JOHN E. MACKENZIE, D.Sc.

COUNCILLORS.

PROFESSOR LANCELOT T. HOGBEN, M.A., D.Sc., F.R.S.

PROFESSOR JAMES RITCHIE, M.A., D.Sc.

G. W. TYRRELL, A.R.C.S., D.Sc., F.G.S.

EMERITUS PROFESSOR C. T. R. WILSON, C.H., M.A., D.Sc., LL.D., F.R.S.

PROFESSOR R. C. GARRY, M.B., Ch.B., D.Sc.

PROFESSOR D. MURRAY LYON, M.D., F.R.C.P.E., D.P.H., D.Sc.

J. E. RICHEY, M.C., B.A., Sc.D., F.R.S., F.G.S.

THE HON. LORD ROBERTSON.

A. GRAHAM DONALD, M.A., F.F.A., F.S.A.Scot.

ALAN W. GREENWOOD, D.Sc., Ph.D.

EMERITUS PROFESSOR T. H. MILROY, M.D., LL.D.

W. P. D. WIGHTMAN, Ph.D., M.Sc.

OFFICE STAFF.

Assistant Secretary and Librarian, G. A. STEWART.
Assistant Librarian, R. J. B. MUNRO.
Housekeeper, W. BRYCE.
Tel. No. 22881.

PATRON.

HIS MOST EXCELLENT MAJESTY THE KING.

FELLOWS OF THE SOCIETY,

Corrected to 23rd October 1939.

N.B.—*Those marked* * *are Annual Contributors.*
" " † *have commuted Voluntary Contribution (see 3rd Paragraph, Law VI).*

M-B. prefixed to a name indicates that the Fellow has received a Makdougall-Brisbane Medal.
K. " " " " Keith Medal.
N. " " " " Neill Medal.
V. J. " " " " the Gunning Victoria Jubilee Prize.
B. " " " " Bruce Medal.
B-P. " " " " Bruce-Preller Lectureship.
C. " " " has contributed one or more Communications to the Society's TRANSACTIONS or PROCEEDINGS.

Date of Election			Service on Council, etc.
1925	M-B. C.	* Aitken, Alexander Craig, M.A., D.Sc., F.R.S. (SECRETARY TO ORDINARY MEETINGS), Lecturer in Actuarial Science, University of Edinburgh (Drummond Street). 54 Braid Road, Edinburgh 10	1934–36. Sec. 1936–
1889		† Alison, John, M.A., LL.D., formerly Head Master, George Watson's College. 126 Craiglea Drive, Edinburgh 10	
1927	C.	* Allan, Douglas Alexander, D.Sc., Director, City of Liverpool Public Museums, William Brown Street, Liverpool	
1920	C. M-B.	* Allen, Herbert Stanley, M.A. (Cantab.), D.Sc. (Lond.), F.R.S., Professor of Natural Philosophy, University of St Andrews	1921–24.
1939		* Amies, Arthur Barton Pilgrim, D.D.Sc. (Melb.), L.D.S. (Vict.), L.R.C.P.E., L.R.C.S.E., D.L.O. (Melb.), F.R.A.C.S., Professor of Dental Science, Australian College of Dentistry. 193 Spring Street, Melbourne	
1938		* Anderson, Charles Henry William Gatacre, B.Sc. (Edin.), Headmaster and Superintendent, Royal Blind School, Edinburgh. 12 West Savile Road, Edinburgh 9	
1920	C. M-B.	* Anderson, Ernest Masson, M.A., D.Sc., F.G.S., 62 Greenbank Crescent, Edinburgh 10	
1905		Anderson, William, M.A., formerly Head Science Master, George Watson's College, Edinburgh. 6 Lockharton Crescent, Edinburgh 11	
1905		Andrew, George, C.B.E., M.A., B.A., H.M.I.S. (retired). Hamewith, Kilmacolm, Renfrewshire	
1930		* Annan, William, M.A., C.A., Professor of Accounting and Business Method, University of Edinburgh (South Bridge). Toftkill, Ferry Road West, Edinburgh 5	
1939		* Annan, William Gillies, M.D. (Edin.), F.R.C.S.E., Clinical Medical Officer, Durham County Council. 41 Cleveland Avenue, Darlington	
1915		Anthony, Charles, M.Inst.C.E., M.Am.Soc.C.E., F.R.San.I., F.R.Met.S., F.R.A.S., F.C.S., Springcroft, Les Croutes, St Peter Port, Guernsey, Channel Islands	
1906		Appleton, Colonel Arthur Frederick, F.R.C.V.S. (*no permanent address until further notice*)	
1910	C.	Archibald, E. H., B.Sc., Professor of Chemistry, University of British Columbia, Vancouver, Canada	
1933		* Arnot, Frederick Latham, Sc.D., Ph.D. (Cantab.), B.Sc. (Sydney), Lecturer in Physics, University of Sydney, N.S.W.	
1921		* Arthur, William, M.A., Lecturer in Mathematics, University of Glasgow. 148 Carmunnock Road, Cathcart, Glasgow	
1938		* Bacsich, Paul, M.D. (Szeged, Hungary), Lecturer in Human Embryology, University of Glasgow. 34 Victoria Crescent Road, Dowanhill, Glasgow, W. 2	
1920		* Bagnall, Richard Siddoway, Hon. D.Sc., F.R.E.S., 3 St Helen's Terrace, Low Fell, Co. Durham	
1920	C. N.	* Bailey, Edward Battersby, M.C., M.A., D.Sc. (Harvard and Birmingham), F.R.S., F.G.S., Director, Geological Survey of Great Britain and Museum of Practical Geology, Exhibition Road, London, S.W. 7	1932–35. V-P 1935–37.

Date of Election			Service on Council, etc.
1896	C.	† Baily, Francis Gibson, M.A., M.Inst.E.E., Emeritus Professor of Electrical Engineering, Heriot-Watt College, Edinburgh. Newbury, Juniper Green, Midlothian	1909–12, 1920–23. V.P 1929–32.
1934		* Bain, David, M.Sc. (Manch.), D.Sc. (Edin.), Lecturer in Technical Chemistry, University of Edinburgh (West Mains Road). 87 Cluny Gardens, Edinburgh 10	
1931	C.	* Bain, William Alexander, Ph.D., Reader in Pharmacology, School of Medicine, University of Leeds. 26 Weetwood Road, Headingley, Leeds 6	
1931		* Baird, Sir William Macdonald, Kt., J.P., Fellow and Past President of the Faculty of Surveyors of Scotland, F.S.A.Scot. Dalveen, Barnton Avenue, Davidson's Mains, Edinburgh 4	
1921		* Baker, Bevan Braithwaite, M.A., D.Sc., Professor of Mathematics, Royal Holloway College, Englefield Green, London	
1928	C.	* Baker, Edwin Arthur, D.Sc. (Edin.), Assistant at the Royal Observatory, Edinburgh. 17 Ladysmith Road, Edinburgh 9	
1905	C.	Balfour-Browne, William Alexander Francis, M.A., F.Z.S., F.L.S., F.R.E.S., F.R.M.S., Barrister-at-Law, formerly Professor of Entomology, Imperial College of Science, London. Hook Place, Burgess Hill, Sussex	
1933		* Banerjee, Prabodh Chandra, L.R.C.P.E., L.R.C.S.E., F.R.F.P.S.G., F.A.C.S., Lt.-Col., I.M.S. C/o Lloyds Bank, Ltd., 101–1 Clive Street, Calcutta, India	
1938		* Bannerman, David Armitage, M.B.E., Sc.D., M.A. (Cantab.), Supernumerary Staff, Department of Zoology, British Museum (Nat. Hist.). 7 Pembroke Gardens, Kensington, London, W. 8	
1928		Barbour, George Brown, M.A. (Edin.), M.A. (Cantab.), Ph.D., F.G.S., Department of Geology, University of Cincinnati, Ohio, U.S.A.	
1886		Barclay, A. J. Gunion, M.A., 44A Belsize Park Gardens, Hampstead, London, N.W. 3	
1903		Bardswell, Noël Dean, M.V.O., M.D., F.R.C.P. (Lond.), M.R.C.P.E. (*no permanent address until further notice*)	
1929		* Barker, Sydney George, O.B.E., D.I.C., F.Inst.P., Scientific Advisor, Indian Jute Mills Association. 191 Coombe Lane, Wimbledon, London, S.W. 20	
1914	C.	Barkla, Charles Glover, M.A., D.Sc., F.R.S., Professor of Natural Philosophy, University of Edinburgh (Drummond Street), Nobel Laureate, Physics, 1917. 4 Church Hill, Edinburgh 10	1915–18, 1924–27.
1937		* Barnett, Adam John Guilbert, B.Sc., Ph.D. (Edin.), Lecturer in Chemistry, Education Department, Lagos, Nigeria, B.W.A.	
1927		* Barnett, John, F.F.A., C.A., Scottish Widows' Fund Life Assurance Society, 9 St Andrew Square, Edinburgh 2	
1921		* Bartholomew, John, M.C., M.A., F.R.G.S., Geographical Institute, Duncan Street, Edinburgh. The Manor House, Inveresk	1925–28.
1927		* Bastow, Stephen Everard, M.Inst.E.E., M.Inst.Min.E. Northwood, Russell Place, Trinity, Edinburgh 5	
1929		* Bath, Frederick, Ph.D., Lecturer in Mathematics, University of Edinburgh (Drummond Street)	
1939		* Baxter, Rev. James Houston, D.D. (Glas.), D.Litt. (St Andrews), Regius Professor of Ecclesiastical History, University of St Andrews, St Mary's College. 71 South Street, St Andrews	
1936		* Bayliss, Leonard Ernest, B.A., Ph.D., Lecturer in Biophysics, University of Edinburgh (Teviot Place). 52 Palmerston Place, Edinburgh 12	
1913		† Beard, Joseph, F.R.C.S.E., M.R.C.S., L.R.C.P. (Lond.), D.P.H. (Cantab.), formerly Medical Officer of Health, City of Carlisle. 8 Carlton Gardens, Carlisle	
1888		Beare, Sir Thomas Hudson, Kt., J.P., D.L., B.A., B.Sc., LL.D. (Edin.), M.Inst.C.E., Hon. M.I.Mech.E., Professor of Engineering, University of Edinburgh (Sanderson Engineering Laboratory, Mayfield Road). 10 Regent Terrace, Edinburgh 7	1907–09. V-P 1909–15, 1923–26.
1897	C.	Beattie, Sir John Carruthers, K.B., D.Sc., LL.D., formerly Vice-Chancellor and Principal, The University, Cape Town	
1893	C. M-B.	Becker, Ludwig, Ph.D., Emeritus Regius Professor of Astronomy, University of Glasgow. The Observatory, University Gardens, Glasgow	
1939		* Beevers, Cecil Arnold, D.Sc. (Liverpool), F.Inst.P., Dewar Fellow in Crystallography, University of Edinburgh (Chemistry Department, West Mains Road)	

Date of Election			Service on Council, etc.
1933		* Begg, James Livingstone, F.G.S. (Treasurer, Geological Society of Glasgow). Elms, Mount Vernon, Glasgow	
1916		* Bell, Robert John Tainsh, M.A., D.Sc., LL.D. (Glas.), Professor of Mathematics, University of Otago, Dunedin, New Zealand	
1929		* Bennet, George, A.H.-W.C., B.Sc., A.M.I.Mech.E., Lecturer in Mechanical Engineering, Heriot-Watt College. 68 Arden Street, Edinburgh 10	
1893	C.	Berry, Sir George A., M.B., C.M., LL.D., F.R.C.S.E., King's Knoll, North Berwick	1916–19. V-P 1919–22.
1936		Berry, John, M.A. (Cantab.), Ph.D. (St Andrews), Superintendent, River Forss Salmon Research, Caithness (Fisheries Division, Scottish Home Department). Tayfield, Newport, Fife	
1897	C.	Berry, Richard J. A., M.D., F.R.C.S.E., Director of Medical Services, Stoke Park Colony, Stapleton, Bristol. Rufford, Canford Lane, Westbury-on-Trym, Bristol	
1938		* Beveridge, Alexander William Morton, formerly Treasurer of the Bank of Scotland, and *ex officio* Chairman of the Managers of the Banks in Scotland. 44 Inverleith Place, Edinburgh 4	
1932		Bhatia, Sohan Lal, M.C., M.A., M.D. (Cantab.), F.R.C.P. (Lond.), Major, I.M.S., Professor of Physiology and Dean, Grant Medical College, Bombay. Two Gables, Mount Pleasant Road, Malabar Hill, Bombay, India	
1937		* Biswas, Kalipada, M.A., D.Sc.(Edin.), Superintendent Royal Botanic Garden, Calcutta; Lloyd Botanic Garden, Darjeeling, and Calcutta Gardens, India	
1937		* Blackie, Joseph John, Ph.D. (Edin.), F.C.S., F.I.C., a Partner in Messrs Duncan, Flockhart & Co., Chemical Manufacturers, 104 Holyrood Road, Edinburgh 8	
1936		* Blair, Duncan MacCallum, M.B., Ch.B. (Glas.), F.R.F.P.S.G., D.Sc. (Lond.), Regius Professor of Anatomy, University of Glasgow. 2 The University, Glasgow, W. 2	
1933		* Bolam, Thomas Robert, M.Sc. (Bristol), D.Sc. (Edin.), Lecturer in Chemistry, University of Edinburgh (West Mains Road). 8 Wilton Road, Edinburgh 9	
1915		Boon, Alfred Archibald, D.Sc., B.A., F.I.C., Emeritus Professor of Chemistry, Heriot-Watt College, Ednburgh. 87 Warrender Park Road, Edinburgh 9	
1937	C.	* Born, Max, M.A. (Cantab.), Hon. D.Sc. (Bristol), Dr.phil. (Göttingen), F.R.S., Tait Professor of Natural Philosophy, University of Edinburgh (Drummond Street). 84 Grange Loan, Edinburgh 9	
1925		* Bose, Sahay Ram, M.A., D.Sc., F.N.I. (India), Professor of Botany, Carmichael Medical College, Belgachia, Calcutta, India	
1886	C. N.	Bower, Frederick Orpen, M.A., D.Sc., LL.D., F.R.S., F.L.S., Emeritus Regius Professor of Botany, University of Glasgow. 2 The Crescent, Ripon, Yorks	1887–90, 1893–96, 1907–09, 1917–19. V-P 1910–16. P 1919–24.
1926		* Braid, Kenneth William, M.A. (Cantab.), B.Sc., Professor of Botany, West of Scotland Agricultural College, 6 Blythswood Square, Glasgow	
1907		† Bramwell, Edwin, M.D., LL.D., F.R.C.P. (Edin. and Lond.), formerly Professor of Clinical Medicine, University of Edinburgh. 23 Drumsheugh Gardens, Edinburgh 3	1935–38.
1932		* Brash, James Couper, M.C., M.A., M.B., Ch.B. (Edin.), M.D. (Birm.), Professor of Anatomy, University of Edinburgh (Teviot Place)	1932–35.
1918		* Bremner, Alexander, M.A., D.Sc., formerly Headmaster, Demonstration School, Training Centre, Aberdeen. 13 Belgrave Terrace, Aberdeen	
1893		Brock, G. Sandison, M.B.E., M.D., F.R.C.P.E., Greenbanks, St Saviour's, Jersey, Channel Islands	
1938		* Brook, Geoffrey Bernard, D.Sc. (Vet. Sci.), F.R.C.V.S., Health of Animals Division, Ministry of Agriculture, London, S.W. 1	
1937		* Brook, George Bernard, F.I.C., F.C.S., formerly Chief Chemist to the British Aluminium Co., Ltd., London. Erracht, Roslin, Midlothian	
1934		* Brough, Patrick, M.A., B.Sc. (Glas.), D.Sc. (Sydney), Lecturer in Botany, University of Sydney, N.S.W.	
1907		Brown, Alexander, M.A., B.Sc., Professor of Applied Mathematics, University, Cape Town	

Date of Election			Service on Council, etc.
1936		* Brown, Andrew Johnstone, F.R.C.S.E., L.R.C.P.E., L.D.S. (Edin.), Lecturer on Metallurgy, Tutor in charge of Anæsthetic Department and Demonstrator in Radiology, Edinburgh Dental Hospital and School. 51 Minto Street, Edinburgh 9	
1937		* Brown, Archibald Gray Robertson, F.F.A., Manager and Actuary of the Life Association of Scotland. Barnshalloch, 30 Gillespie Road, Colinton, Edinburgh 13	
1928		* Brown, Hugh Wylie, F.I.A., F.F.A., 1 Cobden Crescent, Edinburgh 9	
1924	C.	* Brown, Thomas Arnold, M.A., B.Sc., Professor of Mathematics, University College, Exeter	
1923		* Brown, Walter, M.A., B.Sc., Professor of Mathematics, University, Hong Kong, China	
1935		* Brownlie, James Law, M.D. (Glas.), D.P.H., M.R.C.P.E., formerly Chief Medical Officer, Department of Health for Scotland. C/o Mr Ross, Rannoch, 17 Ophir Road, Bournemouth, Hants	
1921		* Bruce, Alexander, B.Sc. (Edin.), Government Agricultural Analyst and City Analyst for Colombo, Kandy and Galle, The Laboratory, Turret Road W., Colombo, Ceylon	
1912		Bruce, Alexander Ninian, D.Sc., M.D., 8 Ainslie Place, Edinburgh 3	
1936		* Bruce, Sir Robert, Kt., D.L., J.P., LL.D., formerly Editor of the *Glasgow Herald*. Brisbane House, 9 Rowan Road, Glasgow, S. 1	
1898	C. K.	† Bryce, Thomas Hastie, M.A., M.D. (Edin.), LL.D., F.R.S., Emeritus Professor of Anatomy, University of Glasgow. The Loaning, Peebles	1911–14, 1922–25, 1935–38. V.-P. 1925–28.
1936		* Bryden, William, M.Sc., B.A., Ph.D. (Edin.), Union House, The University, Melbourne, S. Australia	
1917		* Burnside, George Barnhill, M.Inst.Mech.E., Fairhill, Dullatur	
1930	C.	* Burt, David Raitt Robertson, B.Sc. (St Andrews), F.L.S., Professor of Zoology, Ceylon University College, Colombo	
1896		Butters, John W., M.A., B.Sc., formerly Rector of Ardrossan Academy. 116 Comiston Drive, Edinburgh 10	
1929	C.	* Calder, Alexander, Ph.D., Chief Marketing Officer, Pig Marketing Board, Thames House, Millbank, London, S.W. 1	
1910		Calderwood, Rev. Robert Sibbald, D.D., formerly Minister of Cambuslang. 84 Findhorn Place, Edinburgh 9	
1893	C.	Calderwood, William Leadbetter, I.S.O., formerly Inspector of Salmon Fisheries of Scotland. Ardnacoille, Carr Bridge, Inverness-shire	1923–26.
1926	C. M-B.	* Cameron, Alfred Ernest, M.A., D.Sc. (Aberd.), Steven Lecturer in Agricultural and Forest Zoology, University of Edinburgh (10 George Square). 8 West Savile Road, Edinburgh 9	
1933		* Cameron, Finlay James, F.F.A., F.I.A., General Manager, Caledonian Insurance Company. Beech Knowe, Barnton, near Edinburgh	
1905	C.	Cameron, John, M.D., D.Sc., M.R.C.S. (Eng.), formerly Professor of Anatomy, Dalhousie University, Halifax, Nova Scotia. Balmashanner, Grove Road, East Cliff, Bournemouth	
1921		* Campbell, Andrew, Advisory Chemist, c/o Burmah Oil Co., Ltd., Research Laboratory, Fairlawn, Honor Oak Road, Forrest Hill, London, S.E. 30 Foxgrove Road, Beckenham, Kent	
1918		* Campbell, John Menzies, D.D.S. (Toronto), L.D.S. (Glas.), L.D.S. (Ontario), F.I.C.D., 14 Buckingham Terrace, Glasgow, W.	
1915	C. N.	Campbell, Robert, M.A., D.Sc., F.G.S., Reader in Petrology, University of Edinburgh (Grant Institute of Geology, West Mains Road). Maryton, Colinton	1920–23.
1927	C.	* Cannon, Herbert Graham, M.A., Sc.D. (Cantab.), D.Sc. (Lond.), M.Sc. (Manc.), F.R.S., F.L.S., Beyer Professor of Zoology, University of Manchester. Hollin Knowle, Chapel-en-le-Frith, Derbyshire	
1899	C.	Carlier, Edmund W. W., B. ès Sc., M.Sc., M.D., F.R.E.S., Emeritus Professor of Physiology, University of Birmingham. Morningside, Dorridge, near Birmingham	
1938		* Carlow, Charles Augustus, M.Inst.C.E., M.I.M.E., Managing Director, Fife Coal Co., Ltd. Linnwood Hall, Leven, Fife	
1910		Carnegie, Col. David, C.B.E., J.P., M.Inst.C.E., The Haven, Seasalter, Whitstable	

Date of Election			Service on Council, etc.
1931		* Carroll, John Anthony, M.A., Ph.D. (Cantab.), Professor of Natural Philosophy, University of Aberdeen. Marischal College, Aberdeen	
1905	C.	Carse, George Alexander, M.A., D.Sc., Reader in Natural Philosophy, University of Edinburgh (Drummond Street). 3 Middleby Street, Edinburgh 9	
1901		Carslaw, Horatio Scott, M.A., Sc.D. (Cantab.), D.Sc., LL.D. (Glas.), Emeritus Professor of Mathematics, University of Sydney, New South Wales. Burradoo, New South Wales	
1933		* Carswell, John Irvine, B.Sc., Ph.D., A.M.Inst.C.E., A.M.Inst.Mech.E., Lecturer in Engineering, University of Edinburgh (Sanderson Engineering Laboratory, Mayfield Road). 43 Mansionhouse Road, Edinburgh 9	
1925		* Carter, George Stuart, M.A., Ph.D., Corpus Christi College, Cambridge	
1932		* Cathcart, Edward Provan, C.B.E., M.D., D.Sc., LL.D., F.R.S., Professor of Physiology, University of Glasgow. 28 Hillhead Street, Glasgow	
1939		* Chakravarti, Amulyaratan, M.B., B.Sc. (Calcutta), M.R.C.P.E., Research Fellow and Consulting Biochemist, B.I. Research Laboratories, Calcutta. 1 Furriapooker Street, Calcutta, India	
1932	C.	* Childe, Vere Gordon, B.A., B.Litt., D.Litt. (Harvard), Hon. D.Sc. (Pennsylvania), F.R.A.I., F.S.A., Professor of Prehistoric Archæology, University of Edinburgh	
1925	C.	* Chumley, James, M.A., Ph.D., formerly Lecturer in Oceanography, Department of Zoology, University of Glasgow. Thalassa, Milton Road East, Portobello, Midlothian	
1928	C.	* Clark, Alfred Joseph, M.C., B.A., M.D., F.R.S., Professor of Materia Medica, University of Edinburgh (Teviot Place). 67 Braid Avenue, Edinburgh 10	1932–35.
1933		* Clark, Arthur Melville, M.A. (Edin.), D. Phil. (Oxon.), Lecturer in English Literature, University of Edinburgh. 34 Bruntsfield Gardens, Edinburgh 10	
1891		Clark, John Brown, C.B.E., J.P., M.A., LL.D., formerly Head Master of George Heriot's School. Garleffin, 146 Craiglea Drive, Edinburgh 10	1928–1931. V.P 1931–34.
1935		* Clark, Robert Selbie, M.A., D.Sc. (Aberd.), Scientific Superintendent (Fisheries Division, Scottish Home Department). The Cottage, Murtle, Aberdeenshire	
1932		* Clark, Sir Thomas, Bart., Publisher, Head of T. & T. Clark, Ltd. 6 Wester Coates Road, Edinburgh 12	
1909		Clayton, Thomas Morrison, M.D., D.Hy., B.Sc., D.P.H., Medical Officer of Health, Greenesfield House, Gateshead-on-Tyne	
1932		* Clouston, David, C.I.E., M.A., B.Sc. (Agric.), D.Sc., F.R.S.G.S., formerly Director, Imperial Agricultural Research Institute, Pusa. Forthview, Boswall Road, Edinburgh 5	
1936	C.	* Cockburn, Alexander Murray, Ph.D. (Edin.), Assistant, Geological Department, University of Edinburgh (Grant Institute of Geology, West Mains Road). 53 Ladysmith Road, Edinburgh 9	
1904	C.	Coker, Ernest George, M.A. (Cantab.), D.Sc. (Edin.), Hon. D.Sc. (Sydney and Louvain), M.Sc. (McGill), F.R.S., M.Inst.C.E., M.I.Mech.E., Emeritus Professor of Civil and Mechanical Engineering, University of London. Engineering Laboratories, 3 Farnley Road, Chingford, London, E. 4	
1904		Coles, Alfred Charles, M.D., D.Sc., York House, Poole Road, Bournemouth, W.	
1888	C. V. J.	Collie, John Norman, Ph.D., D.Sc., LL.D., F.R.S., F.C.S., F.I.C., Emeritus Professor of Organic Chemistry, University College, Gower Street, London. 20 Gower Street, London, W.C. 1	
1938		* Colquhoun, Sir Iain, K.T., Baronet of Colquhoun and Luss, D.S.O., LL.D., Rector, University of Glasgow, 1934–37. Rossdu, Luss	
1909	C.	Comrie, Peter, M.A., B.Sc., LL.D., formerly Rector, Leith Academy. 19 Craighouse Terrace, Edinburgh 10	
1936		* Cooper, The Rt. Hon. Thomas Mackay, O.B.E., K.C., M.P., M.A., LL.B., His Majesty's Advocate for Scotland. 7 Abercromby Place, Edinburgh 3	
1924	C.	* Copson, Edward Thomas, M.A. (Oxon.), D.Sc. (Edin.), Professor of Mathematics, University College, Dundee (University of St Andrews). 14 Balmyle Road, Broughty Ferry, Dundee	
1937		* Cousland, Charles Johnstone, A.R.P.S., formerly President, Royal Scottish Society of Arts. Achray, Kinnear Road, Edinburgh 4	
1928		* Coutie, Rev. Alexander, B.Sc., Ph.D., Minister, Church of Scotland, Watten, Caithness	

Date of Election			Service on Council, etc.
1914		Coutts, William Barron, M.A., B.Sc., F.Inst.P., Assistant Professor of Fire Control Instruments, Military College of Science, Woolwich, S.E. 18. Craigmillar, 70 Cambridge Drive, London, S.E. 12	
1911		Cowan, Alexander, M.A. (Cantab.), Papermaker, Valleyfield, Penicuik, Midlothian	
1931		* Cowan, John Macqueen, M.A., D.Sc. (Edin.), B.A. (Oxon.), F.L.S., Assistant Keeper, Royal Botanic Garden, Edinburgh. 17 Inverleith Place, Edinburgh 4	
1935		* Cowan, Samuel Hunter, Lt.-Col. R. E., Lecturer in Forestry Engineering, University of Edinburgh (George Square). New Club, Edinburgh 2	
1916	C.	† Craig, E. H. Cunningham, B.A. (Cantab.), Geologist and Mining Engineer, The Dutch House, Beaconsfield	
1908		Craig, James Ireland, M.A., B.A., 88 Sharia Kasr el Eini, Cairo, Egypt	
1925	C. K.	* Craig, Robert Meldrum, M.A., D.Sc., F.G.S., Lecturer in Economic Geology, University of Edinburgh (Grant Institute of Geology, West Mains Road)	
1937		* Craig, William Stuart McRae, B.Sc. (Glas.), M.D. (Edin.), F.R.C.P.E., Medical Officer, Ministry of Health, Whitehall, London, S.W. 1. Mellendean, Harriots Lane, Ashtead, Surrey	
1933		* Craig-Bennett, Arthur Lancelot, M.A., Ph.D. (Cantab.), formerly Chief Fisheries Officer, Haifa, Palestine. 1 High Street, Musselburgh	
1903		† Crawford, Lawrence, M.A. (Cantab.), D.Sc. (Glas.), LL.D. (Wit.), formerly Professor of Pure Mathematics, University, Cape Town. 21 Pillans Road, Rosebank, Cape Town	
1922	C.	* Crew, Francis Albert Eley, M.D., D.Sc., Ph.D., F.R.S., Professor of Animal Genetics and Director of the Institute of Animal Genetics, University of Edinburgh (West Mains Road). 41 Mansionhouse Road, Edinburgh 9	1928–31. Sec. 1931–36. V-P 1936–39.
1931		* Crichton, John, M.A., B.Sc. (Edin.), 17 Willows Avenue, Morden, Surrey	
1936		* Cronshaw, Cecil John Turrell, B.Sc. (Vict.), Hon. D.Sc. (Leeds), F.I.C., Chairman, Dyestuffs Group of Imperial Chemical Industries, Ltd. (including Scottish Dyes, Ltd.), Hexagon House, Blackley, Manchester	
1929		* Cruickshank, Ernest William Henderson, M.D., D.Sc., Ph.D., Regius Professor *of Physiology, University of Aberdeen*	
1938		* Cruickshank, Martin Melvin, M.D., Ch.M. (Aberd.), F.R.C.S.E., D.O.M.S. (Lond.), Lieut.-Col. I.M.S., and Professor of Surgery, Madras Medical College. Pantheon House, Pantheon Road, Madras	
1914		† Cumming, Alexander Charles, O.B.E., D.Sc., Red Latches, King's Drive, Caldy, Cheshire	
1928		* Cumming, William Murdoch, D.Sc. (Glas.), F.I.C., M.Inst.Chem.E., "Young" Professor of Technical Chemistry, Royal Technical College, Glasgow. Bonnieblink, 4 Newlands Road, Newlands, Glasgow, C. 1	
1917		* Cunningham, Brysson, D.Sc., B.E., M.Inst.C.E., Editor *The Dock and Harbour Authority*, formerly Lecturer in Waterways, Harbours, and Docks, University College, London. 141 Copers Cope Road, Beckenham, Kent	
1930		* Cunningham, John, C.I.E., B.A., M.D., Lt.-Colonel, I.M.S. (retired). South Bank, Grange Loan, Edinburgh 10	
1938		* Cursiter, Stanley, O.B.E., R.S.W., R.S.A., Director of National Galleries of Scotland, Brunstane House, Portobello, Midlothian	
1934		* Daly, Ivan de Burgh, M.A., M.D., B.Ch., Professor of Physiology, University of Edinburgh (Teviot Place). Cooliney, Spylaw Avenue, Edinburgh	1935–38.
1934		* Darling, Frank Fraser, Ph.D. (Edin.), N.D.A., Isle of Tanera, Achiltibuie, Garve, Ross and Cromarty	
1924	V. J.	* Darwin, Charles Galton, M.C., M.A., Sc.D., LL.D., F.R.S., formerly Tait Professor of Natural Philosophy, University of Edinburgh, and Master of Christ's College, Cambridge. Director, National Physical Laboratory, Teddington, Middlesex. Bushy House, Teddington, Middlesex	1925–28. Sec. 1928–33. V-P 1933–36.
1935	C.	* Davidson, Charles Findlay, B.Sc. (St Andrews), F.G.S., Geologist, H.M. Geological Survey of Great Britain, Geological Survey and Museum, Exhibition Road, South Kensington, London, S.W. 7	
1939		* Davidson, James, F.S.A.Scot., Treasurer of the Carnegie Trust for the Universities of Scotland. 59 Morningside Park, Edinburgh 10	
1932		* Davidson, Leybourne Stanley Patrick, B.A. (Cantab.), M.D. (Edin.), F.R.C.P.E., Professor of Medicine, University of Edinburgh (Teviot Place)	

Date of Election			Service on Council, etc.
1935		* Davidson, Maxwell, B.Sc. Eng., Ph.D., M.I.Mech.E., Lecturer in Heat Engines and Thermodynamics, University of Edinburgh (Sanderson Engineering Laboratory, Mayfield Road). 47 Liberton Drive, Edinburgh 9	
1939		* Davies, Francis, M.D., D.Sc. (Lond.), B.S., M.R.C.S. (Eng.), L.R.C.P. (Lond.), Arthur Jackson Professor of Anatomy, University of Sheffield. 5 Tapton House Road, Sheffield, 10	
1930	C.	* Davies, Lewis Merson, M.A., Ph.D., F.G.S., F.R.A.I., Lt.-Colonel, Royal Artillery (retired). 8 Garscube Terrace, Murrayfield, Edinburgh 12	
1928		Dawson, Warren Royal, F.R.S.L., F.S.A., F.S.A.Scot., Honorary Librarian of Lloyd's, London, E.C. 3. Simpson House, Simpson, Bletchley, Bucks	
1923		* Deane, Arthur, M.R.I.A., Curator, Public Art Gallery and Museum, Belfast. Beach Road, Whitehead, Belfast	
1935	C.	* Desai, Bhimbhai Nichhabhai, B.A., M.Sc. (Bombay), D.Sc., Ph.D. (Edin.), Assistant Meteorologist, Government of India, Meteorological Office, Poona 5, India	
1938		* Dewar, John Michael, M.D. (Edin.), F.R.C.P.E. 5 Chalmers Street, Edinburgh 3	
1937		* Dhar, Sasindra Chandra, D.Sc. (Edin.), Head of the Department of Mathematics, College of Science, Nagpur, C.P., India	
1924		* Dinham, C. H., B.A., H.M. Geological Survey. Edgemoor, 19 Highfield Road, Northwood, Middlesex	
1923		* Dixon, Ronald Audley Martineau, of Thearne, F.G.S., F.S.A.Scot., F.R.G.S., Thearne Hall, near Beverley	
1881	C.	Dobbin, Leonard, Ph.D. (VICE-PRESIDENT), formerly Reader in Chemistry, University of Edinburgh. Faladam, Blackshiels, Midlothian	1904–07, 1913–16. Curator 1934–39. V-P 1939–
1918		* Dodd, Alexander Scott, Ph.D., F.I.C., F.C.S., City Analyst for Edinburgh. 20 Stafford Street, Edinburgh 3	
1902		Dollar, John A. W., F.R.C.V.S., 72 Maida Vale, London, W. 9	
1925		* Donald, Alexander Graham, M.A., F.F.A., F.S.A.Scot., Secretary of the Scottish Provident Institution, Edinburgh. 18 Carlton Terrace, Edinburgh 7	1939–
1937	C.	* Donald, Hugh Paterson, M.Agr.Sc. (N.Z.), D.Sc. (Edin.), Lecturer, Institute of Animal Genetics, University of Edinburgh (West Mains Road). Hill Cottage, West Mains Road, Edinburgh 9	
1882	C.	Dott, David B., F.I.C., Memb. Pharm. Soc., Ravenslea, Musselburgh	
1938		* Dott, Norman McOmish, F.R.C.S.E., Neurological Surgeon to the Royal Infirmary and the Lecturer on Neurological Surgery, University of Edinburgh. 3 Chalmers Crescent, Edinburgh 9	
1921	C. M-B.	* Dougall, John, M.A., D.Sc., 47 Airthrey Avenue, Glasgow, W. 4	
1901 & 1918		Douglas, Carstairs Cumming, M.D., D.Sc., Professor of Medical Jurisprudence and Hygiene, Anderson's College, Glasgow. Clonbeith, Milngavie, by Glasgow	
1910		Douglas, Loudon MacQueen, Newpark, Mid-Calder, Midlothian	
1934		* Dow, David Rutherford, M.D., D.P.H., F.R.C.P.E., Professor of Anatomy, University of St Andrews (University College, Dundee). 16 Windsor Street, Dundee	
1932		* Drennan, Alexander Murray, M.D. (Edin.), F.R.C.P.E., Professor of Pathology, University of Edinburgh (Teviot Place)	
1923	C.	* Drever, James, M.A., B.Sc., D.Phil., Professor of Psychology, University of Edinburgh (South Bridge). Ivybank, Wardie Road, Edinburgh 5	1929–32.
1901		Drinkwater, Thomas W., L.R.C.P.E., L.R.C.S.E., Chemical Laboratory, Surgeons' Hall, Edinburgh	
1923		* Drummond, J. Montagu F., M.A. (Cantab.), Harrison Professor of Botany, University of Manchester	1928–31.
1925		* Dryerre, Henry, M.R.C.S. (Eng.), Ph.D., Professor of Physiology, Royal (Dick) Veterinary College; Physiological Biochemist, Animal Diseases Research Association. Kenmore, Lasswade	
1921		* Drysdale, Charles Vickery, C.B., O.B.E., D.Sc. (Lond.), M.I.E.E., F.Inst.P., formerly Director of Scientific Research to the Admiralty. 151 Wick Hall, Furge Hill, Hove, Sussex	

Date of Election			Service on Council, etc.
1937		* Dunlop, Derrick Melville, B.A. (Oxon.), M.D. (Edin.), F.R.C.P.E., Christison Professor of Therapeutics, University of Edinburgh (Teviot Place). 1 Moray Place, Edinburgh 3	
1904		Dunlop, William Brown, M.A., 4A St Andrew Square, Edinburgh 2. Seton Castle, Longniddry, E. Lothian	
1933		* Dymond, Edmund Gilbert, M.A., Lecturer in Natural Philosophy, University of Edinburgh (Drummond Street). 7 Greenhill Gardens, Edinburgh 10	
1934		* Edge, William Leonard, M.A., Sc.D. (Cantab.), Lecturer in Mathematics, University of Edinburgh (Drummond Street)	
1931		* Eggleton, Philip, D.Sc., Lecturer in Biochemistry, Department of Physiology, University of Edinburgh (Teviot Place)	
1924		* Elliot, Right Hon. Walter Elliot, P.C., M.C., M.B., Ch.B., D.Sc., LL.D., M.P., F.R.S., Minister of Health. 60 Eaton Square, London, S.W. 1	
1938		* Elphinstone, The Right Hon. Lord, K.T., LL.D., formerly President, Royal Scottish Geographical Society. Carberry Tower, Musselburgh, Midlothian	
1933		* Erskine, John Maxwell, General Manager of the Commercial Bank of Scotland, Ltd. Hazlieburn, West Linton	
1934	C.	* Etherington, Ivor Malcolm Haddon, B.A. (Oxon.), Ph.D. (Edin.), Lecturer in Technical Mathematics, University of Edinburgh (Drummond Street). 41 Scotland Street, Edinburgh 3	
1924		* Evans, Arthur Humble, M.A., Sc.D., Lecturer in English History. Cheviot House, Crowthorne, Berks	
1924		Evans, William Edgar, B.Sc., Assistant in charge of Herbarium, Royal Botanic Garden, Edinburgh 4	
1902		Ewen, John Taylor, O.B.E., J.P., B.Sc., M.I.Mech.E., H.M. Inspector of Schools (Emeritus), Pitscandly, Forfar	
1939		* Eyles, Victor Ambrose, B.Sc. (Bristol), Senior Geologist, H.M. Geological Survey (Scotland), 19 Grange Terrace, Edinburgh 9. 27 Mentone Terrace, Edinburgh 9	
1900	C.	Eyre, John William Henry, M.D., M.S. (Dunelm), D.P.H. (Cantab.), Emeritus Professor of Bacteriology, Guy's Hospital. 51 Portland Place, London, W.1	
1931		* Fairbairn, William Ronald Dodds, M.A., M.D., D.Psych. (Edin.), F.R.A.I. 18 Lansdowne Crescent, Edinburgh 12	
1936		* Fairweather, James Falconer, W.S., N.P., Fiscal of the Society of Writers to the Signet. 14 Henderland Road, Edinburgh 12	
1907	C.	Falconer, John Downie, M.A., D.Sc., F.G.S., formerly Director of the Geological Survey of Nigeria. The Cedars, Hatton Road, Feltham, Middlesex	
1923		* Feldman, William Moses, M.D., B.S., F.R.C.P. (Lond.), F.R.A.S., Senior Physician, St Mary's Hospital for Women and Children, Plaistow. 851 Finchley Road, London, N.W. 11	
1928		* Fenton, Edward Wyllie, M.A., D.Sc. (Aberd.), F.L.S., Head of Botany Department, Edinburgh and East of Scotland College of Agriculture, 13 George Square, Edinburgh 8	
1907		Fergus, Edward Oswald, c/o Messrs M'Kay & Boyd, Solicitors, 50 Wellington Street, Glasgow	
1933		* Ferguson, Thomas, M.D., D.P.H. (Edin.), F.R.C.P.E., D.Sc., Medical Officer, Department of Health for Scotland. 32 Learmonth Street, Falkirk	
1925	C.	* Ferrar, William Leonard, M.A., Fellow and Tutor of Hertford College, Oxford	
1932		* Findlay, Sir John Edmund Ritchie, Bart., B.A. (Oxon.), M.P., Proprietor of the *Scotsman*. Aberlour House, Aberlour, Banffshire	
1927	C.	* Finlay, Thomas Matthew, M.A., D.Sc. (Edin.), Lecturer in Palæontology, University of Edinburgh (Grant Institute of Geology, West Mains Road). 11 Dudley Terrace, Leith 6	
1911		† Fleming, John Arnold, F.C.S., Pottery Manufacturer, Locksley, Helensburgh, Dumbartonshire	
1906		Fleming, Robert Alexander, M.A., M.D., LL.D., F.R.C.P.E., Consulting Physician, Royal Infirmary. 10 Chester Street, Edinburgh 3	
1900	C. N.	Flett, Sir John S., K.B.E., M.A., D.Sc., LL.D., F.R.S., formerly Director of the Geological Survey of Great Britain and Museum of Practical Geology, Exhibition Road, South Kensington, London, S.W. 7	1916–19.
1892		Ford, John Simpson, F.I.C., 7 Corrennie Drive, Edinburgh 10	
1928	C.	* Forrest, James, M.A., B.Sc. (Glas.), D.Sc. (St Andrews), Lecturer in Physics, University College, Dundee. Cumbrae, Oxford Street, Blackness, Dundee	

Date of Election			Service on Council, etc.
1933		* Forrester, Charles, A.H.-W.C., Ph.D., F.I.C., F.Inst.F., Principal and Professor of Chemistry, Indian School of Mines, Dhanbad, India	
1920	C.	* Franklin, Thomas Bedford, M.A. (Cantab.), 28 Kingshill Drive, Kenton, Harrow, Middlesex	
1910		Fraser, Alexander, Actuary, 5 St Margaret's Road, Edinburgh 9	
1929		* Fraser, David Kennedy, M.A., B.Sc., Psychologist to Glasgow Education Authority. Edge o' the Moor, Milngavie, Dumbartonshire	
1934		* Fraser, George, Chartered Civil Engineer, M.Inst.C.E., M.I.Struct.E. 25 Murrayfield Gardens, Edinburgh 12	
1939		* Fraser, Ian, M.D., M.Ch., F.R.C.S. (Eng.), Consultant Surgeon. 33 Wellington Park, Belfast	
1928		* Fraser, Sir John, K.C.V.O., M.C., M.D., Ch.M., F.R.C.S.E., Regius Professor of Clinical Surgery, University of Edinburgh (Royal Infirmary). 20 Moray Place, Edinburgh 3	
1928		* Fraser, Kenneth, M.D. (Edin.), D.P.H. (Cantab.), D.T.M. (Edin.), County Medical Officer of Health, Cumberland. The Croft, Scotby, near Carlisle	
1914		Fraser, William, F.R.A.S., F.S.A.Scot., Managing Director, Neill & Co., Ltd., Printers, 212 Causewayside, Edinburgh 9	
1907		Galbraith, Alexander M., 6 Broomhill Avenue, Glasgow, W. 1	
1933		* Galbraith, Augustus William de Rohan, M.Inst.C.E., M.Inst.C.E.I., F.S.E., City Engineer, Christchurch, New Zealand, Chairman of the Advisory Council of the New Zealand Standard Institute. The Spur, Clifton, Sumner, New Zealand	
1939		* Galloway, John Galloway, J.P., F.I.S.I., Master of the Edinburgh Merchant Company. 7 Stirling Road, Edinburgh 5	
1901		Ganguli, Sanjiban, M.A., formerly Principal, Maharaja's College, and Director of Public Instruction, Jaipur State, Hathroi-Garh Road, Jaipur (Rajputana), India	
1933		* Gardner, Alfred Charles, M.Inst.C.E., M.Inst.Mech.E., M.Inst.E.E., F.G.S., Chief Engineer, Clyde Navigation Trust, 16 Robertson Street, Glasgow. 117 Fotheringay Road, Glasgow, S. 1	
1926		* Gardner, John Davidson, B.Sc., A.M.Inst.C.E., Partner, Gardner & Oswald, Civil Engineers, 84 George Street, Edinburgh 2	
1937		* Garry, Robert Campbell, M.B., Ch.B., D.Sc. (Glas.), Professor of Physiology, University College (University of St Andrews), Dundee. 58 Seafield Road, Broughty Ferry, Dundee	1938–
1930	C.	* Geddes, Alexander Ebenezer M'Lean, O.B.E., M.A., D.Sc., Lecturer in Natural Philosophy, University of Aberdeen. 12 Louisville Avenue, Aberdeen	
1909	C.	† Geddes, Rt. Hon. Sir Auckland Campbell, P.C., G.C.M.G., K.C.B., M.D., LL.D., Frensham, The Layne, Rolvenden, Kent	
1909		† Gentle, William, B.Sc., Head Master, George Heriot's School. 10 West Savile Road, Edinburgh 9	
1939		* Ghosh, Birendra Nath, M.B.E., F.R.F.P.S.G., L.M. (Dub.), Professor of Pharmacology, Carmichael Medical College, Calcutta. 9 Taltolla Lane, Calcutta, India	
1914		Gibb, Sir Alexander, G.B.E., C.B., LL.D., F.R.S., M.Inst.C.E., Queen Anne's Lodge, Westminster, London, S.W. 1	
1910	C.	Gibb, David, M.A., B.Sc., Reader in Mathematics, University of Edinburgh (Drummond Street). 45 Fountainhall Road, Edinburgh 9	
1936		* Gibbons, Sydney Guy, Ph.D. (Lond.), Naturalist, Fishery Board for Scotland, Aberdeen. Dunmore, Milltimber, Aberdeenshire	
1917	C.	* Gibson, Alexander, M.B., Ch.B., F.R.C.S. (Eng.), 620 Medical Arts Building, Winnipeg, Canada	
1921		* Gibson, Walcot, D.Sc., F.R.S., F.G.S., *formerly* Assistant Director, H.M. Geological Survey (Scotland). Pathways, Fairlight Road, Hythe, Kent	
1911		Gidney, Sir Henry A. J., Kt., J.P., M.L.A., F.R.C.S.E., Lt.-Col., I.M.S. (retired), c/o The Allahabad Bank, Ltd., Calcutta, India	
1938		* Gifford, Charles Henry Pearson, M.A. (Cantab.), Partner in Baillie, Gifford & Co., 3 Glenfinlas Street, Edinburgh 3. 32 Stafford Street, Edinburgh 3	
1937		* Gilchrist, Andrew Rae, M.D. (Edin.), F.R.C.P.E., M.R.C.P. (Lond.), Assistant Physician, Royal Infirmary, Edinburgh, and Lecturer on Therapeutics, University of Edinburgh (Teviot Place). 6 Lansdowne Crescent, Edinburgh 12	

Date of Election			Service on Council, etc.
1933		* Gillespie, Robert Pollock, M.A., B.Sc., Ph.D. (Cantab.), Lecturer in Mathematics, University of Glasgow, 51 Clouston Street, Glasgow, N.W.	
1933		* Gillespie, Thomas Haining, Director-Secretary, Zoological Society of Scotland. Corstorphine Hill House, Murrayfield, Edinburgh 12	
1925		* Gillies, William King, M.A., B.A., F.E.I.S., LL.D. (Glas.), Rector of the Royal High School, Edinburgh. Davaar, 12 Suffolk Road, Edinburgh 9	
1909		Gladstone, Hugh Steuart, M.A., M.B.O.U., F.Z.S., Capenoch, Penpont, Dumfriesshire	
1911		Gladstone, Reginald John, M.D., F.R.C.S. (Eng.), Lecturer and Senior Demonstrator of Anatomy, King's College, University of London. 22 Court Lane Gardens, London, S.E. 21	
1934		* Glaister, John, M.D., D.Sc. (Glas.), Professor of Forensic Medicine, University of Glasgow. 43 Victoria Crescent Road, Glasgow, W. 2	
1925	C.	* Goldie, Archibald Hayman Robertson, M.A., B.A., D.Sc., Assistant Director, Meteorological Office, Air Ministry, Kingsway, London, W.C. 2. 28 Burghley Road, Wimbledon, London, S.W. 19	1929–32.
1901		Goodwillie, James, M.A., B.Sc., 239 Clifton Road, Aberdeen	
1938		* Gordon, Cecil, M.Sc. (Cape Town), Ph.D. (Lond.), Lecturer in Genetics, Department of Natural History, University of Aberdeen	
1938		* Gordon, William Smith, Ph.D., M.R.C.V.S., Chief Bacteriologist, Moredun Institute, Animal Diseases Research Association, Gilmerton. 2 Northfield, Liberton, Edinburgh	
1913	C. M-B.	Gordon, William Thomas, M.A., D.Sc. (Edin.), M.A. (Cantab.), Professor of Geology, University of London, King's College, Strand, W.C.	
1897		Gordon-Munn, John Gordon, M.D., Croys, Castle Douglas	
1934		* Gorrie, Robert Maclagan, D.Sc. (Edin.), Silviculturalist to Punjab Government. C/o Chief Conservator of Forests, Lahore, Punjab, India	
1938		* Goyal, Ram Kumar, M.B., B.S. (Punjab), M.R.C.P. (Lond.), M.R.C.S. (Eng.), Ph.D. (Edin.), Research Worker, School of Tropical Medicine, Calcutta, India	
1923		* Grabham, George Walter, O.B.E., M.A. (Cantab.), F.G.S., Government Geologist, Anglo-Egyptian Sudan. Box 178, Khartoum	
1924		Graham, Robert James Douglas, M.A., D.Sc. (SECRETARY TO ORDINARY MEETINGS), Professor of Botany, University of St Andrews	1938–39. Sec. 1939–
1931		* Grant, Robert, J.P., Publisher (Oliver & Boyd), Edinburgh. 6 Kilgraston Road, Edinburgh 9	
1935	C.	* Grant, Ronald, Ph.D. (Edin.), Department of Zoology, McGill University, Montreal, Canada	
1918		* Gray, William Forbes, F.S.A.Scot., 8 Mansionhouse Road, Edinburgh 9	
1939		* Greaves, William Michael Herbert, M.A. (Cantab.), F.R.A.S., Astronomer Royal for Scotland, Professor of Astronomy, University of Edinburgh. Royal Observatory, Blackford Hill, Edinburgh	
1937	C.	* Green, George, M.A., D.Sc. (Glas.), Lecturer in Applied Physics, University of Glasgow. 64 Partickhill Road, Glasgow, W. 1	
1938		* Greenlees, James Robertson Campbell, M.A., M.B., B.C. (Cantab.), Headmaster of Loretto School, Musselburgh	
1927	C. K.	* Greenwood, Alan William, D.Sc. (Melb.), Ph.D. (Edin.), Lecturer in the Institute of Animal Genetics, University of Edinburgh (West Mains Road)	1939–
1922		* Greenwood, Rev. William Osborne, M.D. (Leeds), B.S. (Lond.), L.S.A., Clerk in Holy Orders, Kirklands, Slingsby Walk, Harrogate, Yorks	
1906		Greig, Edward David Wilson, C.I.E., M.D., D.Sc., F.R.C.P.E., Lt.-Col., I.M.S. (retired), 38 Coates Gardens, Edinburgh 12	
1931		* Greig, John Russell, Ph.D. (Edin.), Director, Moredun Institute, Animal Diseases Research Association. Wedderlie, Kirkbrae, Liberton 9	
1905		† Greig, Sir Robert Blyth, M.C., LL.D., formerly Secretary to the Department of Agriculture for Scotland. The Shaws, Barnton, Midlothian	1921–24. V.P. 1924–27.
1935		* Grierson, Alexander Millar Meek, M.D. (Edin.), D.P.H. (Edin. and Glas.), Senior Assistant to the Medical Officer of Health, Public Health Department, Sunlight House, Manchester 3	
1910		Grimshaw, Percy Hall, I.S.O., F.R.E.S., formerly Keeper, Natural History Department, Royal Scottish Museum. 133 Liberton Brae, Edinburgh 9. (*Died 14th November* 1939)	
1899		† Guest, Edward Graham, J.P., M.A., B.Sc., 5 Newbattle Terrace, Edinburgh 10	

Date of Election			Service on Council, etc.
1927		* Gulland, John Masson, M.A. (Oxon.), D.Sc. (Edin.), Ph.D. (St Andrews), Sir Jesse Boot Professor of Chemistry, University College, Nottingham	
1907		Gulliver, Gilbert Henry, D.Sc., A.M.I.Mech.E., 99 Southwark Street, London, S.E.	
1930		* Guthrie, Douglas, M.D., F.R.C.S.E., Lecturer in Diseases of the Ear, Nose, and Throat, School of Medicine of the Royal Colleges, Edinburgh. 21 Clarendon Crescent, Edinburgh 4	
1933	C.	* Guthrie, William Gilmour, M.A. (Edin.), B.A. (Cantab.), Ph.D., Professor of Mathematics and Natural Philosophy, Magee College, Londonderry, Northern Ireland	
1911		Guy, William, F.R.C.S.E., L.R.C.P.E., L.D.S.Ed., LL.D. (Penn.), Consulting Dental Surgeon, Edinburgh Royal Infirmary; Lecturer on Human and Comparative Dental Anatomy and Physiology. 11 Wemyss Place, Edinburgh 3	
1934		* Haldane, David, B.Sc. (Edin.), Senior Geologist, H.M. Geological Survey (Scotland), 19 Grange Terrace, Edinburgh 9. 6 Kilmaurs Road, Edinburgh 9	
1939		* Hall, William Robert, J.P., Manager, Union Bank of Scotland, Ltd. (Head Office), 64 George Street, Edinburgh 2. The Sheiling, Barnton, Midlothian	
1936	C. N.	* Hamilton, William James, D.Sc. (Glas.), M.D., B.Ch. (Belfast), Professor of Anatomy, St Bartholomew's Hospital Medical College, London, E.C. 1. *Temporary Address*, Anatomy Department, Downing Street, Cambridge	
1922		* Hannay, Robert Kerr, M.A., LL.D., H.R.S.A., Fraser Professor of Scottish History and Palæography, University of Edinburgh (South Bridge). Historiographer-Royal for Scotland. 5 Royal Terrace, Edinburgh 7	
1923		* Hanneford-Smith, William, A.M.Inst.C.E., Hon. A.R.I.B.A. 1 The Avenue, Gravesend, Kent	
1918		* Hardie, Patrick Sinclair, M.A., B.Sc., formerly Head of the Physics Department, Medical School, Cairo. 19 Ardmillan Terrace, Edinburgh 11	
1928		* Harding, William Gerald, F.R.Hist.S., F.S.A.Scot., F.R.E.S., Peckwater House, Charing, Kent	
1939		* Harper, Walter Fearn, M.D., Ph.D. (St Andrews), Reader in Anatomy, University of London, Department of Anatomy, London Hospital Medical College, Turner Street, London, E. 1	
1923	C.	* Harris, Robert Graham, M.A., D.Sc. (Edin.), 44 Manor Road, Farnborough, Hants	
1914		Harrison, Edward Philip, Ph.D., F.Inst.P., Chief Scientist, H.M.S. "Vernon," Portsmouth	
1934		Harrison, John Vernon, D.Sc.(Glas.), F.G.S., 34 Rowallan Gardens, Glasgow, W. 1	
1921		* Harrison, John William Heslop, D.Sc. (Durham), F.R.S., Professor of Botany and Reader in Genetics, King's College, Newcastle-upon-Tyne. The Avenue, Birtley, Co. Durham	
1926		* Harvey, William Frederick, C.I.E., M.A., M.B., C.M., D.P.H., Lieut.-Col., I.M.S. (retired), *Histologist, Research Laboratory, Royal College of* Physicians, Edinburgh. 56 Garscube Terrace, Edinburgh 12	
1936		* Henderson, David Kennedy, M.D., F.R.C.P.E., F.R.F.P.S.G., Professor of Psychiatry, University of Edinburgh. Tipperlinn House, Edinburgh 10	
1931		* Henderson, John, F.C.I.I., Manager and Secretary, Edinburgh Assurance Co., Ltd. Seaforth Cottage, York Road, Trinity, Edinburgh 5	
1936		* Henderson, Thomas, B.Sc. (Lond.), Secretary to the Educational Institute of Scotland. 2 Hillview Terrace, Edinburgh 12	
1929		* Henderson, Thomas, C.B.E., J.P., F.S.A.Scot., Actuary of the Savings Bank of Glasgow. 5 Belmont Crescent, Glasgow, W. 2	
1908		Henderson, William Dawson, M.A., B.Sc., Ph.D., Lecturer, Zoological Laboratories, University, Bristol. 77 Coldharbour Road, Bristol 6	
1937		* Hepburn, William Allan Forsyth, M.C., M.A., B.Ed. (Edin.), Director of Education, Ayrshire Education Authority. 2 St Leonard's Road, Ayr	
1925		* Heron, Alexander Macmillan, D.Sc. (Edin.), F.G.S., formerly Director, Geological Survey of India, Calcutta, India	
1916		* Herring, Percy Theodore, M.D., F.R.C.P.E., Professor of Physiology, University of St Andrews. Linton, St Andrews	1917–20, 1931–34. V-P 1934–37.

Date of Election			Service on Council, etc.
1936		* Hewat, Andrew Fergus, M.D., F.R.C.P.E., Secretary, Royal College of Physicians, Edinburgh. 14 Chester Street, Edinburgh 3	
1922		Hindle, Edward, M.A., Sc.D. (Cantab.), Ph.D., A.R.C.S., Regius Professor of Zoology, University of Glasgow	
1928	C.	* Hobson, Alfred Dennis, M.A. (Cantab.), Professor of Zoology, King's College, Newcastle-upon-Tyne	
1928		* Hodge, William Vallance Douglas, M.A. (Edin.), M.A. (Cantab.), F.R.S., Lowndean Professor of Astronomy and Geometry, University of Cambridge. 28 Barrow Road, Cambridge	
1938		* Hogarth, George, Chairman of the Herring Industry Board, Fisheries Division, Scottish Home Department, St Andrew's House, Edinburgh. 40 Elliot Road, Edinburgh 11	
1923	C. K.	* Hogben, Lancelot Thomas, M.A., D.Sc., F.R.S., Professor of Natural History, University of Aberdeen	1937–
1927		Holden, Henry Smith, D.Sc., F.L.S., Director of the East Midlands Forensic Science Laboratory, Burton Street, Nottingham	
1930		* Holland, Sir Thomas Henry, K.C.S.I., K.C.I.E., D.L., Hon. D.Sc., LL.D., F.R.S., Vice-Chancellor and Principal of the University of Edinburgh. Blackford Brae, Edinburgh 9	1931–32. V-P 1932–35.
1929	C.	* Hora, Sunder Lal, D.Sc. (Punjab et Edin.), F.L.S., F.Z.S., F.A.S.B., Senior Assistant Superintendent, Zoological Survey of India. Indian Museum, Calcutta	
1920	C.	* Horne, Alexander Robert, O.B.E., B.Sc., M.I.Mech.E., A.M.Inst.C.E., Professor of Mechanical Engineering, Heriot-Watt College, Edinburgh. 31 Queen's Crescent, Edinburgh 9	
1896		Horne, John Fletcher, M.D., F.R.C.S.E., Shelley Hall, Huddersfield	
1912	C. M-B.	† Houstoun, Robert Alexander, M.A., Ph.D., D.Sc., F.Inst.P., Lecturer in Physical Optics, University of Glasgow. 45 Kirklee Road, Glasgow, W.2	1929–32.
1893		Howden, Robert, M.A., M.B., C.M., D.Sc., LL.D., Emeritus Professor of Anatomy, University of Durham. Dalruadh, Victoria Terrace, Crieff	
1933		* Hume, Edgar Erskine, D.S.M., M.A., M.D., LL.D., Lieut.-Col., U.S. Army, Librarian of the Army Medical Library, Washington. The Magnolias, Frankfort, Kentucky. *Temporary Address*, Medical Field Service School, Carlisle Barracks, Pa., U.S.A.	
1910		Hume, William Fraser, D.Sc. (Lond.), Director, Geological Survey of Egypt, Helwân, Egypt. The Laurels, Rustington, Sussex	
1927		* Hunt, Owen Duke, B.Sc. (Manch.), Corrofell, Newton Ferrers, South Devon	
1923		* Hunter, Rev. Adam Mitchell, M.A., D.Litt., Librarian of New College, Edinburgh. 3 Suffolk Road, Edinburgh 9	
1932		* Hunter, Andrew, M.A., B.Sc., M.B., Ch.B., F.R.S.C., F.R.F.P.S.G., Professor of Pathological Chemistry, University of Toronto, Canada	
1928		* Hunter, Arthur, F.F.A., LL.D. (Edin.), Chevalier de la Légion d'Honneur, Vice-President and Chief Actuary of the New York Life Insurance Co. 124 Lloyd Road, Montclair, N.J., U.S.A.	
1916		* Hunter, Charles Stewart, M.A., L.R.C.P.E., L.R.C.S.E., D.P.H., F.R.E.S., Cotswold, 36 Streatham Hill, London, S.W. 2	
1911		Hunter, Gilbert Macintyre, M.Inst.C.E., M.Inst.E.S., M.Inst.M.E., 27 Kilmaurs Road, Edinburgh 9. (*Died 7th December* 1939)	
1935		* Hutchinson, Arthur Cyril William, M.D.S. (Manch.), D.D.S. (Witwatersrand), Dean of the Edinburgh Dental Hospital and School. 12 Glencairn Crescent, Edinburgh 12	
1923	C.	* Ince, Edward Lindsay, M.A. (Cantab.), D.Sc. (Edin.), Lecturer in Technical Mathematics, University of Edinburgh (Drummond Street)	
1927		* Inglis, John Alexander, of Auchindinny and Redhall, K.C., M.A. (Oxon.), LL.B. (Edin.), King's and Lord Treasurer's Remembrancer (VICE-PRESIDENT), Auchindinny House, Milton Bridge, Midlothian	1935–38. V-P 1939–
1912		Inglis, Robert John Mathieson, M.Inst.C.E., Chief Engineer, Southern Area, L.N.E.R. Dixton, Monken Hadley, Herts	
1936		* Innes, Donald Esme, M.C., M.A. (Oxon.), Professor of Geology, University of St Andrews. Cortina, St Andrews	
1917		* Irvine, Sir James Colquhoun, Kt., C.B.E., D.L., Ph.D. (Leipzig), D.Sc. (St Andrews), Hon. D.Sc. (Liverpool, Princeton), Hon. Sc.D. (Cantab., Yale, Pennsylvania), Hon. LL.D. (Glas., Aberd., Edin., and Toronto), Hon. D.C.L. (Durham), F.R.S., Hon. F.E.I.S., Vice-Chancellor and Principal of the University of St Andrews	1920–22. V-P 1922–25.

Date of Election			Service on Council, etc.
1930	C.	* Jack, David, M.A., B.Sc. (Edin.), Ph.D. (St Andrews), Lecturer in Natural Philosophy, United College, University of St Andrews. 22 Grange Road, St Andrews	
1923		* Jack, John Louttit, C.B.E., Solicitor, Deputy Secretary, Department of Health for Scotland, St Andrew's House, Edinburgh	
1938		* Jamieson, James Dalgleish Hamilton, H.D.D., L.D.S.Ed., Lecturer on Dental Diseases, University of Edinburgh. 29 and 58 George Square, Edinburgh 8	
1912	C.	Jeffrey, George Rutherford, M.D. (Glas.), F.R.C.P.E., 11 Langland Gardens, Hampstead, London, N.W. 3	
1934		* Jeffrey, Sir John, K.C.B., C.B.E., formerly Under-Secretary of State for Scotland. 9 Cluny Gardens, Edinburgh 10	
1906	C. K.	Jehu, Thomas John, M.A., M.D., F.G.S., Professor of Geology, University of Edinburgh (Grant Institute of Geology, West Mains Road)	1917–20, 1923–26. V-P 1929–32.
1900		† Jerdan, David Smiles, M.A., D.Sc., Ph.D., Avenel, Melrose	
1939		* Johnson, Norman Miller, B.Sc. (Manch.), F.E.I.S., F.R.S.G.S., Fellow Bot. Soc. Edin., Headmaster of the Commercial Public School, Dunfermline. 115 Victoria Terrace, Dunfermline	
1936		* Johnston, John McQueen, M.D. (Glas.), F.R.C.S.E., Pharmacologist, Department of Health for Scotland. 18 Berkeley Terrace, Glasgow, C. 3	
1928		* Johnston-Saint, Percy Johnston, M.A. (Cantab.), Conservator, Wellcome Historical Medical Museum, 183-193 Euston Road, London, N.W. 1. 4 Wyndham Place, Bryanston Square, London, W. 1	
1928		* Johnstone, Robert William, C.B.E., M.A., M.D. (Edin.), F.R.C.S.E., M.R.C.P.E., Professor of Midwifery and Diseases of Women, University of Edinburgh. 26 Palmerston Place, Edinburgh 12	
1927		* Jones, Edward Taylor, D.Sc. (Lond.), Hon. D.Sc. (Wales), Professor of Natural Philosophy, University of Glasgow	1927–30.
1928	C.	* Jones, Tudor Jenkyn, Sc.D., M.D. (Glas.), Lecturer in Anatomy (Embryology), University of Liverpool. 49 Prince Aldred Road, Liverpool 15	
1922		* Juritz, Charles Frederick, M.A., D.Sc., F.I.C., Chief of the Union Department of Chemistry. Grenoble, Avenue Fresnaye, Sea Point, Cape Town, South Africa.	
1936		* Kemball, Charles Henry, H.D.D., L.D.S., D.D.S. (Univ. Pennsylvania), Dental Surgeon, Lecturer on Orthodontics in Edinburgh Dental Hospital and School. 20 Ainslie Place, Edinburgh 3	
1925	C.	* Kemp, Charles Norman, B.Sc., Technical Radiologist, Secretary of the Royal Scottish Society of Arts. Ivy Lodge, Laverockbank Road, Edinburgh 5	
1929	C.	* Kendall, James Pickering, M.A., D.Sc., F.R.S. (GENERAL SECRETARY), Professor of Chemistry, University of Edinburgh (West Mains Road). 14 Mayfield Gardens, Edinburgh 9	1931–33. Sec. 1933–36. Gen. Sec. 1936–
1912		† Kennedy, Robert Foster, M.D. (Belfast), M.B., B.Ch. (R.U.I.), Associate Professor of Neurology, Cornell University, New York. 410 East 57 Street, New York City, U.S.A.	
1927		* Kennedy, Walter Phillips, Ph.D. (Edin.), L.R.C.P. and S.E., A.I.C., Medical Officer in the Ministry of Health, Whitehall, London, S.W. 1. 112 Beaufort Street, Chelsea, London, S.W. 3	
1935		* Kenneth, John Henry, M.A., Ph.D. (Edin.), Assistant, Imperial Bureau of Animal Genetics, University of Edinburgh (West Mains Road). University Union, Edinburgh 8	
1909		Kenwood, Henry Richard, C.M.G., M.B., C.M., Emeritus Chadwick Professor of Hygiene, University of London. Wadhurst, Queen's Road, Finsbury Park, London, N.	
1925	C. M-B.	* Kermack, William Ogilvy, M.A., D.Sc., LL.D., Chemist, Research Laboratory of the Royal College of Physicians, 2 Forrest Road, Edinburgh 1	
1903 & 1923	C. N.	Kerr, Sir John Graham, Kt., M.A. (Cantab.), LL.D., F.R.S., Honorary Fellow of Christ's College, M.P., Scottish Universities, Emeritus Regius Professor of Zoology, University of Glasgow. Dalny Veed, Barley, near Royston, Herts	1904–07, 1913–16, 1924–27. V-P 1928–31.
1891		Kerr, Joshua Law, M.D., J.P. Stratford, Victoria, Australia	

Date of Election			Service on Council, etc.
1926		* Khastgir, Satis Ranjan, M.Sc. (Calcutta), D.Sc. (Edin.), Physics Department, University, Dacca, India	
1907		King, Archibald, M.A., B.Sc., H.M. Inspector of Schools, The Cottage, Barassie, Ayrshire	
1925		* King, Leonard Augustus Lucas, M.A., Professor of Zoology, West of Scotland Agricultural College, Glasgow. 14 Bank Street, Glasgow, W. 2	
1918		* Kingon, Rev. John Robert Lewis, M.A., D.Sc., The Manse, Simonston, Cape Colony, South Africa	
1937		* Kirby, Percival Robson, M.A., D.Litt., F.R.C.M., Professor of Music and Musical History, University of the Witwatersrand, Johannesburg, South Africa	
1937	C.	* Koller, Peo Charles, Ph.D. (Budapest), D.Sc. (Edin.), Cytologist, Institute of Animal Genetics, University of Edinburgh (West Mains Road). 169 Mayfield Road, Edinburgh 9	
1927		* Lambie, Charles George, M.C., M.D., F.R.C.P.E., Bosch Professor of Medicine, University of Sydney. Capri, 4 Wyuna Road, Point Piper, Sydney, N.S.W., Australia	
1920	C.	* Lamont, John Charles, Lieut.-Col., I.M.S. (retired), C.I.E., M.B., C.M. (Edin.), M.R.C.S. (Eng.), formerly Professor of Anatomy, Medical College, Lahore, India. 7 Merchiston Park, Edinburgh 10	
1939		* Landale, Stenard Ernest Andrew, Ph.D. (Cantab.), D.Phil. (Oxon.), Honours Diploma, Faraday House, London, Director of William Younger & Co. and of Scottish Brewers, Ltd. Ellersly, Murrayfield, Edinburgh 12, and Cranshaws, Duns, Berwickshire	
1925	C. N.	* Lang, William Henry, M.B., C.M., D.Sc., LL.D. (Glas.), F.R.S., Barker Professor of Cryptogamic Botany, University of Manchester	
1931		* Langrishe, John du Plessis, D.S.O., M.B., B.Ch. (Dub.), D.P.H., Lt.-Col. R.A.M.C. (retired), Lecturer in Public Health, University of Edinburgh (Usher Institute of Public Health, Warrender Park Road). 2 South Gillsland Road, Edinburgh 10	
1910	C.	Lauder, Alexander, D.Sc., formerly Head of Chemistry Department, Edinburgh and East of Scotland College of Agriculture, and Lecturer in Agricultural Chemistry, University of Edinburgh. 78 Dalkeith Road, Edinburgh 9	1917–20. Sec. 1923–28.
1885	C.	Laurie, Arthur Pillans, M.A., D.Sc., LL.D., formerly Principal, Heriot-Watt College, Edinburgh. The Cottage, Kiln Way, Grayshott, Hindhead, Surrey	1908–11, 1913–16.
1905		Lawson, David, M.A., M.D., L.R.C.P. and S.E., Druimdarroch, Banchory, Kincardineshire	
1903		Leighton, Gerald Rowley, O.B.E., M.D., D.Sc., formerly Medical Officer (Foods), Department of Health for Scotland. Sharston, near Ramsey, Isle of Man	
1937		* Leitch, William Orr, M.Inst.C.E., formerly Chief Engineer and General Manager, Peking Mukden Railway. 1 Gordon Terrace, Edinburgh 9	
1930		* Lelean, Percy Samuel, C.B., C.M.G., F.R.C.S. (Eng.), L.R.C.P. (Lond.), D.P.H., Professor of Public Health, University of Edinburgh (Usher Institute of Public Health, Warrender Park Road). 4 South Lauder Road, Edinburgh 9	
1910		Levie, Alexander, F.R.C.V.S., D.V.S.M., Balmae, Manor Road, Littleover, Derby	
1916	C.	* Levy, Hyman, M.A., D.Sc., Professor of Mathematics, Imperial College of Science and Technology, London, S.W. 7	
1914	C. N.	Lewis, Francis John, D.Sc., F.L.S., Professor of Botany, Egyptian University, Abbissea, Cairo	
1918		* Lidstone, George James, F.F.A., F.I.A., LL.D., formerly Manager and Actuary, Scottish Widows' Fund Life Assurance Society. Hermiston House, Hermiston, Currie, Midlothian	1919–22.
1905		Lightbody, Forrest Hay, 53 Queen Street, Edinburgh 2	
1931		* Lightfoot, Nicholas Morpeth Hutchinson, M.A. (Cantab.), Lecturer in Mathematics, Heriot-Watt College, Edinburgh. 3 Park Gardens, Liberton, Edinburgh 9	
1923		* Lim, Robert Kho Seng, M.B., Ch.B., D.Sc., Peking Union Medical College, Department of Physiology, Peking, China	
1912		Lindsay, John George, M.A.. B.Sc. (Edin.), Rector of Dunfermline High School	

Date of Election			Service on Council, etc.
1920	C.	* Lindsay, Thomas A., M.A., B.Sc., Head Master, Higher Grade School, Bucksburn, Aberdeenshire	
1912		Linlithgow, The Most Honourable the Marquis of, P.C., K.T., G.C.I.E., D.L., LL.D., Viceroy and Governor-General of India. Hopetoun House, South Queensferry	V-P 1934–37.
1903		† Liston, William Glen, C.I.E., M.D., Lt.-Col., I.M.S. (retired), Milburn Tower, Gogar, Corstorphine, Edinburgh 12	
1929		* Little, John Robert, F.C.I.I., F.C.I.S., formerly Manager and Secretary, Century Insurance Co., Ltd. 5 Dalrymple Crescent, Edinburgh 9	
1932		* Lockhart, James Balfour, M.A., B.Sc. (Edin.), Mathematical Master, Edinburgh Academy. Westering, Inverleith Grove, Edinburgh 4	
1926		* Lorraine, Norman Stanley Rees, M.D., D.P.H. (Edin. and Glas.), Medical Officer of Health, Benfleet Urban District. 1 Burlescoombe Leas, Burlescoombe Road, Thorpe Bay, Southend-on-Sea	
1930		* Low, James Wotherspoon, B.Sc., Ph.D., 1 Hamilton Park Avenue, Glasgow, W. 2	
1934		* Low, R. Cranston, M.D., F.R.C.P.E., formerly Lecturer in Dermatology, University of Edinburgh. 37 Oxgangs Road, Fairmilehead, Edinburgh 10	
1939		* Lucas, Cyril Edward, B.Sc. (Lond.), Officer in Charge University College of Hull Oceanographical Laboratory, Sandport Street, Leith. University College, Hull	
1923	C.	* Ludlam, Ernest Bowman, M.A., D.Sc., Lecturer in Chemistry, University of Edinburgh (West Mains Road)	
1923		* Lyford-Pike, James, M.A., B.Sc., Lecturer in Forestry, University of Edinburgh. Rosetta, 56 Kirkbrae, Liberton, Edinburgh 9	
1924		* Lyon, David Murray, M.D., F.R.C.P.E., D.P.H., D.Sc., Professor of Clinical Medicine, University of Edinburgh (Royal Infirmary). Druim, Colinton, Edinburgh 11	1938–
1929		* M'Arthur, Donald Neil, D.Sc., F.I.C., Professor of Agricultural Chemistry, West of Scotland Agricultural College, Glasgow, C. 2. 35 Kersland Street, Glasgow, W. 2	
1921		* M'Arthur, Neil, M.A., B.Sc., Lecturer in Mathematics, University of Glasgow. The Whins, Heathfield Drive, Milngavie	
1926		* M'Bride, James Alexander, B.A. (Roy. Univ., Ireland), B.Sc. (Lond.), formerly Rector of Queen's Park Secondary School, Glasgow. Scottish Liberal Club, Princes Street, Edinburgh 2	
1883		M'Bride, Peter, M.D., F.R.C.P.E., 3 St Peter's Grove, York	
1931	C.	* McCallien, William John, D.Sc. (Glas.), Lecturer in Geology, University of Glasgow. 116 Henderland Road, Bearsden, Dumbartonshire	
1935		* MacCallum, Peter, M.Sc., M.A. (New Zealand), M.B., Ch.B., D.P.H. (Edin.), Professor of Pathology, University of Melbourne. Blackrock House, Blackrock, Victoria, Australia	
1923		* M'Cracken, William, J.P., F.S.I., Englesea House, Crewe	
1931	C.	* M'Crea, William Hunter, M.A., Ph.D. (Cantab.), B.Sc. (Lond.), F.R.A.S., Professor of Mathematics, The Queen's University, Belfast. 61 Ulsterville Avenue, Belfast	
1918		* M'Culloch, Rev. James David, 17 Eldon Street, Greenock	
1920		* M'Donald, Stuart, M.A., M.D., F.R.C.P.E., formerly Professor of Pathology, University of Durham. C/o British Linen Bank, 141 Princes Street, Edinburgh 2	
1928		* MacDonald, Thomas Logie, M.A., B.Sc. (Glas.), F.R.A.S., Director, British Astronomical Association (Lunar Section). 1 Graingerville North, Newcastle-upon-Tyne 4	
1886		Macdonald, William J., M.A., 15 Comiston Drive, Edinburgh 10	
1931		* M'Dougall, John Bowes, M.D. (Glas.), F.R.F.P.S.G., F.R.C.P.E., Medical Director, British Legion Village, Preston Hall, Kent. Preston Hall, Aylesford, Kent	
1901	C.	MacDougall, R. Stewart, M.A., D.Sc., LL.D. (Edin.), Emeritus Professor of Biology, Royal (Dick) Veterinary College, Edinburgh. Ivy Lodge, Gullane, East Lothian	1914–17.
1910		Macewen, Hugh Allen, O.B.E., M.B., Ch.B., D.P.H. (Lond. and Cantab.), Local Government Board, Ministry of Health, Whitehall, London, S.W.	
1888	C.	M'Fadyean, Sir John, Kt., M.B., B.Sc., LL.D., formerly Principal and Professor of Comparative Pathology, Royal Veterinary College, Camden Town, London. Highlands House, Leatherhead	

Date of Election			Service on Council, etc.
1939		* Macfarlane, James Wright, Ph.D.Eng. (Glas.), Designer of Electrical Machinery and Superintendent of Research and Special-purpose Design. Cartbank, Cathcart, Glasgow, S. 4	
1885	C.	† Macfarlane, John M., D.Sc., LL.D., Emeritus Professor of Botany. 427 West Hansberry Street, Germantown, Pa., U.S.A.	
1938		* MacGillivray, Allister Middleton, M.D. (St Andrews), Lecturer in Clinical Ophthalmology, University of St Andrews. 5 Clarendon Terrace, Dundee	
1897		MacGillivray of MacGillivray, Angus, M.D., C.M., D.Sc., LL.D., Chattan Croft, Crail, Fife	
1878		M'Gowan, George, F.I.C., Ph.D., 21 Montpelier Road, Ealing, London, W. 5	
1932		* MacGregor, Archibald Gordon, M.C., D.Sc., Geologist, H.M. Geological Survey (Scotland). 1 Greenbank Terrace, Edinburgh 10	
1939		* MacGregor, John Ian Graham, M.Inst.C.E., Assistant Engineer (Scottish Area) to the London & North Eastern Railway Co. The Hawthorns, 84 Colinton Road, Edinburgh 11	
1922		* Macgregor, Murray, M.A., D.Sc., F.G.S., Assistant Director (Scotland), H.M. Geological Survey, 19 Grange Terrace, Edinburgh 9	1930-33.
1903		M'Intosh, Donald C., M.A., D.Sc., formerly Education Officer, Elgin. Glen-avon, Boat of Garten	
1911		M'Intosh, John William, M.R.C.V.S., Cairngower, Crieff	
1927	C.	* M'Intyre, Donald, M.B.E., M.D. (Glas.), F.R.C.S.E., Royal Samaritan Lecturer in Gynæcology, University of Glasgow. 9 Park Circus, Glasgow, C. 3	
1912	C.	M'Kendrick, Anderson Gray, M.B., D.Sc., F.R.C.P.E., Lt.-Col., I.M.S. (retired), Superintendent, Research Laboratory, Royal College of Physicians, 2 Forrest Road, Edinburgh 1	1924-27. 1933-36. V.P 1936-39.
1914		M'Kendrick, Archibald, J.P., F.R.C.S.E., D.P.H., L.D.S., 12 Rothesay Place, Edinburgh 3	
1900	C.	M'Kendrick, John Soutar, M.D., F.R.F.P.S. (Glas.), 2 Buckingham Terrace, Hillhead, Glasgow	
1910	C.	Mackenzie, Alister, M.A., M.D., D.P.H., Principal Medical Officer and Lecturer in Hygiene, Training Centre, Jordanhill, Glasgow. 22 Queen's Gate, Dowanhill, Glasgow	
1916	C.	* Mackenzie, John E., D.Sc. (CURATOR), Emeritus Reader in Chemistry, University of Edinburgh. 2 Ramsay Garden, Edinburgh 1	1936-39. Curator 1939-
1938		* Mackie, Alexander, B.Sc., Ph.D. (Edin.), Science Master, Trinity Academy. 2 St John's Terrace, Edinburgh 12	
1929	C.	* Mackie, John, M.A., D.Sc., Rector, Leith Academy. 7 York Road, Trinity, Leith 5	
1928		* Mackie, Thomas Jones, M.D., M.R.C.P.E., Professor of Bacteriology, University of Edinburgh (Teviot Place). 22 Mortonhall Road, Edinburgh 9	
1936		* M'Kinlay, Peter Laird, M.D., D.P.H., Medical Officer (Statistics), Department of Health for Scotland. 69 Montrose Street, Clydebank	
1910		MacKinnon, James, M.A., Ph.D., LL.D., Emeritus Professor of Ecclesiastical History, University of Edinburgh. 12 Lygon Road, Edinburgh 9	1933-36.
1904		Mackintosh, Donald James, C.B., M.V.O., D.L., M.B., C.M., LL.D., Superintendent, Western Infirmary, Glasgow. 3 Kirklee Gardens, Glasgow, W. 2	
1939		* Mackintosh, James Macalister, M.A., M.D. (Glas.), D.P.H. (Lond.), Barrister-at-Law (Gray's Inn), Chief Medical Officer, Department of Health for Scotland. 24 Esslemont Road, Edinburgh 9	
1899		Maclean, Sir Ewen John, J.P., D.L., M.D., D.Sc. (Hon.), LL.D., F.R.C.P. (Lond.), Emeritus Professor of Obstetrics and Gynæcology, Welsh National Medical School. 12 Park Place, Cardiff	
1933		* Macleod, James, F.I.C., Manager, Glasgow Corporation Chemical Works Department. 16 Colebrooke Street, Glasgow, W. 2	
1916	C.	* M'Lintock, William Francis Porter, D.Sc. (Edin.), Geological Survey and Museum, Exhibition Road, South Kensington, London, S.W. 7	
1936		* M'Michael, John, M.D., Ch.B., M.R.C.P.E., British Postgraduate Medical School, Ducane Road, London, W. 12	
1923		* Macmillan, Rt. Hon. Lord, P.C., G.C.V.O., LL.D., Minister of Information, 44 Millbank, Westminster, S.W. 1	
1938		* McMillan, William Hutchison, B.Sc. (Glas.), M.I.M.E., Hood Professor of Mining, University of Edinburgh, and Professor of Mining, Heriot-Watt College, Edinburgh. 5 Gordon Terrace, Edinburgh 9	

Date of Election			Service on Council, etc.
1939		* McNee, John William, D.S.O., M.D., D.Sc. (Glas.), F.R.C.P. (Lond.), Regius Professor of Practice of Medicine, University of Glasgow. Ledcameroch House, Bearsden, Glasgow	
1932		* M'Neil, Charles, M.A., M.D. (Edin.), F.R.C.P.E., Professor of Child Life and Health, University of Edinburgh (Royal Edinburgh Hospital for Sick Children). 44 Heriot Row, Edinburgh 3	
1917		* Macpherson, Rev. Hector Copland, M.A., Ph.D., F.R.A.S., Guthrie Memorial U.F. Church. 7 Wardie Crescent, Edinburgh 5	
1921		* M'Quistan, Dougald Black, M.A., B.Sc., Associate-Professor of Natural Philosophy, Royal Technical College, Glasgow. 29 Viewpark Drive, Rutherglen	
1936		* McRae, William, C.I.E., M.A., D.Sc. (Edin.), Late Agricultural Adviser to the Government of India, and Director, Agricultural Research Institute, Pusa. Daramona, Gamekeepers' Road, Barnton, Midlothian	
1921	C.	* MacRobert, Thomas Murray, M.A., D.Sc., Professor of Mathematics, University of Glasgow. 10 The University, Glasgow	1931-34.
1921	C.	* M'Whan, John, M.A. (Glas.), Ph.D. (Gött.), Lecturer in Mathematics, University of Glasgow. 37 Airthrey Avenue, Jordanhill, Glasgow, W. 4	
1927		* Madwar, Mohamed Reda, Ph.D. (Edin.), A.M.Inst.C.E., Director, Helwân Observatory, Egypt	
1898	C.	† Mahâlanobis, S. C., B.Sc. (Edin.), *formerly* Professor of Physiology, University of Calcutta. 90 Park Street, Calcutta	
1938		* Mainland, Donald, M.B., Ch.B., D.Sc. (Edin.), Professor of Anatomy, Forrest Building, Dalhousie University, Halifax, Nova Scotia	
1939		* Mair, William, F.C.S., F.R.G.S., Manufacturing Chemist (retired). 32 Braid Hills Road, Edinburgh 10	
1938		* Malcolm, Charles Alexander, M.A., Ph.D., Librarian, Signet Library. 21 Findhorn Place, Edinburgh 9	
1933		* Malcolm, John, M.D. (Edin.), Professor of Physiology, University of Dunedin, New Zealand, Medical School, King Street, Dunedin	
1908		Mallik, Devendranath, Sc.D., B.A., Principal, Carmichael College, Rungpur, Bengal, India	
1912		† Maloney, William Joseph, M.B.E., M.C., M.D. (Edin.), LL.D., *formerly* Professor of Neurology, Fordham University. Casa del Sale, Newport, Rhode Island, U.S.A.	
1913		Marchant, Rev. Sir James, K.B.E., LL.D., F.R.A.S., F.L.S., Director, National Council for Promotion of Race-Regeneration. Pinegarth, Buccleuch Road, Bournemouth	
1901	C.	Marshall, Francis Hugh Adam, C.B.E., Sc.D., LL.D., F.R.S., Reader in Agricultural Physiology, University of Cambridge. Christ's College, Cambridge	
1920	C.	* Marshall, John, M.A., D.Sc. (St Andrews), B.A. (Cantab.), University Reader in Mathematics, Bedford College, London. Logan House, 123 Torrington Park, London, N. 12	
1931		* Mason, John Huxley, F.R.C.V.S., Government Veterinary Laboratory, Onderstepoort, Pretoria, South Africa	
1913		Masson, George Henry, O.B.E., M.D., D.Sc., F.R.C.P.E., Carradale, Port of Spain, Trinidad, British West Indies	
1898	C. M-B.	Masterman, Arthur Thomas, M.A., D.Sc., F.R.S., formerly Superintending Inspector, H.M. Board of Agriculture and Fisheries. 3 Kedale Road, Seaford	1902-04.
1939		* Matheson, Donald Capell, F.R.C.V.S., D.V.S.M. (Vict.), Professor of Pathology, Bacteriology and Meat Inspection, Royal (Dick) Veterinary College, Edinburgh. 3 Lockharton Crescent, Edinburgh 11	
1911		† Mathews, Gregory Macalister, C.B.E., M.B.O.U., Meadway, St Cross, Winchester, Hants	
1921		* Mathieson, John, F.R.S.G.S., Division Superintendent, Ordnance Survey (retired), 42 East Claremont Street, Edinburgh 7	
1906		Mathieson, Robert, F.C.S., St Serf's, Innerleithen	
1928		* Matthai, George, M.A. (Cantab.), Sc.D., F.Z.S., F.L.S., Professor of Zoology, Government College, Lahore, India	
1924		* Matthews, James Robert, M.A., F.L.S., Regius Professor of Botany, University of Aberdeen, and Keeper of the Cruickshank Botanic Garden	
1938		* Maxwell, Sir John Maxwell Stirling, K.T., of Pollok, Baronet, and of Corrour, D.L., LL.D. Pollok House, Glasgow, S. 3	

Date of Election			Service on Council, etc.
1932		* Maxwell, William, Managing Director of R. & R. Clark, Ltd. 14 South Inverleith Avenue, Edinburgh 4	
1917		* Maylard, A. Ernest, M.B., B.Sc. (Lond.), F.R.F.P.S. (Glas.), Kingsmuir, Peebles	
1922		* Meakins, Jonathan Campbell, M.D., LL.D., F.R.C.P.E., F.R.C.P. (Lond.), Professor of Medicine and Director of the Department of Medicine, McGill University, Montreal, Canada	
1931		* Mears, Frank Charles, A.R.S.A., F.R.I.B.A., 44 Queen Street, Edinburgh 2	
1937	C.	* Melville, Harry Work, Ph.D. (Edin., Cantab.), D.Sc. (Edin.), Fellow of Trinity College, and Assistant Director, Colloid Science Laboratory, Cambridge	
1901	C.	Menzies, Alan W. C., M.A., Ph.D., F.C.S., Professor of Chemistry, Princeton University, Princeton, New Jersey, U.S.A.	
1927		* Menzies, Sir Frederick Norton Kay, K.B.E., M.D., LL.D. (Edin.), F.R.C.P.E., D.P.H. (Lond.), formerly Medical Officer of Health and School Medical Officer, Administrative County of London. 29 Egerton Gardens, London, S.W. 3	
1933	C.	* Menzies, William John Milne, Inspector of Salmon Fisheries of Scotland (Fisheries Division, Scottish Home Department). Caledonian United Service Club, Edinburgh	
1929		* Mercer, Walter, M.B., Ch.B., F.R.C.S.E., Lecturer in Clinical Surgery, University of Edinburgh (Royal Infirmary). 12 Rothesay Terrace, Edinburgh 3	
1917		* Merson, George Fowlie, Manufacturing Technical Chemist, St John's Hill Works, Edinburgh 8. 7 Cumin Place, Edinburgh 9	
1902	C.	† Metzler, William H., A.B., D.Sc., Ph.D., formerly Dean of the New York State College for Teachers, Albany, N.Y., U.S.A. 5003 South Salina Street, Syracuse, N.Y., U.S.A.	
1885	C. M-B.	Mill, Hugh Robert, D.Sc., LL.D., Hill Crest, Dormans Park, E. Grinstead	
1937		* Miller, James, R.S.A., F.R.I.B.A., Randolphfield, Stirling	
1910		Miller, John, M.A., D.Sc., formerly Professor of Mathematics, Royal Technical College. 5 Kelvinside Terrace West, Glasgow, N.W.	
1936		* Miller, The Very Rev. John Harry, C.B.E., M.A., D.D., formerly Principal of St Mary's College, University of St Andrews. St Mary's, St Andrews	
1930		* Miller, William Christopher, M.R.C.V.S., Courtauld Professor of Animal Husbandry, Royal Veterinary College, Camden Town, London, N.W. 1. Blair Hyrne, Hadley Common, Barnet, Herts	
1905		Milne, Archibald, M.A., D.Sc., Deputy Director of Studies, Edinburgh Provincial Training College. 38 Morningside Grove, Edinburgh 10	
1905		† Milne, Christian Hoyer Millar, M.A., D.Litt., formerly Head Master, Daniel Stewart's College. 19 Merchiston Gardens, Edinburgh 10	
1904	C.	Milne, James Robert, D.Sc., Lecturer in Natural Philosophy, University of Edinburgh (Drummond Street). 7 Grosvenor Crescent, Edinburgh 12	
1886		Milne, William, M.A., B.Sc., 70 Beechgrove Terrace, Aberdeen	
1933	C.	Milne-Thomson, Louis Melville, M.A. (Cantab.), F.R.A.S., Professor of Mathematics, Royal Naval College, Greenwich. Gothic House, Maze Hill, London, S.E. 10	
1899		Milroy, Thomas Hugh, M.D., B.Sc., LL.D., formerly Professor of Physiology, Queen's University, Belfast. Woodville, North Berwick	1939–
1889	C. K.	Mitchell, A. Crichton, D.Sc., Hon. Doc. Sc. (Genève), formerly Director of Public Instruction in Travancore, India. 246 Ferry Road, Edinburgh 5. (Society's Representative on Governing Body of Heriot-Watt College)	1915–16, 1930–33. Cur. 1916–26. V-P 1926–29.
1897		† Mitchell, Sir George Arthur, Kt., D.L., J.P., M.A., LL.D., M.I.M.E., 9 Lowther Terrace, Kelvinside, Glasgow	
1900		Mitchell, James, M.A., B.Sc., Islay Lodge, Lochgilphead, Argyll	
1939		* Molony, John Barré de Winton, M.B., Ch.B., D.P.H. (Edin.), F.R.C.S.E., Lt.-Col. I.M.S. (retired), c/o Chartered Bank of India, Australia and China. 38 Bishopsgate, London, E.C.	
1938		* Monteath, Harry Henderson, W.S., B.A., LL.B., Professor of Conveyancing, University of Edinburgh (South Bridge). 16 Palmerston Place, Edinburgh 12	

Date of Election			Service on Council, etc.
1936		* More, Francis, Chartered Accountant, of the firm of Lindsay, Jamieson and Haldane. 12 Albert Terrace, Edinburgh 10	
1896		Morgan, Alexander, O.B.E., M.A., D.Sc., LL.D., formerly Principal, Edinburgh Provincial Training College. 1 Midmar Gardens, Edinburgh 10	
1936		* Morgan, Daniel Owen, M.Sc. (Wales), Ph.D. (Lond.), Lecturer in Helminthology, Department of Zoology, University of Edinburgh (West Mains Road). 15 Mentone Terrace, Edinburgh 9	
1937		* Morison, John, C.I.E., M.B., Ch.B. (Glas.), D.P.H. (Cantab.), Lt.-Col. I.M.S. (retired). Engaged in Research in the Usher Institute of Public Health. 13 Cluny Drive, Edinburgh 10	
1926		* Morris, James Archibald, R.S.A., F.S.A.Scot., Savoy Croft, Ayr	
1919		* Morris, Robert Owen, O.B.E., M.A., M.D., C.M. (Edin.), D.P.H. (Liverpool). King Edward VII Welsh National Memorial Association (Tuberculosis). Hafod-ar-For, Aberdovey, N. Wales	
1892	C.	Morrison, J. T., M.A., D.Sc., Emeritus Professor of Mathematical Physics, University, Stellenbosch, South Africa	
1930		* Morton, Sir James, Kt., LL.D., Governing Director, Scottish Dyes, Ltd. Dalston Hall, near Carlisle	
1901		Moses, O. St John, M.D., D.Sc., F.R.C.S.E., Lt.-Col., I.M.S. (retired), formerly Professor of Medical Jurisprudence, Medical College, Calcutta. 18 Manstone Road, Cricklewood, London, N.W. 2	
1892	C. K.	Mossman, Robert Cockburn, Lacar 4332, Villa Devoto F.C.P., Buenos Aires, Argentina	
1934		* Mowat, Magnus, C.B.E., T.D., M.Inst.C.E., M.I.Mech.E., Brigadier-General, Ebor House, Sheen Gate Gardens, East Sheen, London, S.W. 14	
1935	C.	* Mozley, Walter Alan, B.Sc. (Manitoba), Ph.D. (Edin.), London School of Hygiene and Tropical Medicine, Keppel Street, London, W.C. 1	
1916		* Muir, Sir Robert, Kt., M.A., M.D., F.R.C.P.E., Sc.D., LL.D., F.R.S., Emeritus Professor of Pathology, University of Glasgow. 30 Victoria Crescent Road, Glasgow, W. 2	
1935		* Munnoch, James, F.S.A.Scot., formerly Controller, General Post Office, Edinburgh. 15 Liberton Drive, Edinburgh 9	
1938	C.	* Munro, Sanford Sterling, B.S.A. (McGill), M.S. (Wisconsin), D.Sc. (Edin.), Poultry Geneticist, Dominion Department of Agriculture, Ottawa, Canada	
1939		* Murray, Alastair Campbell, F.F.A., Manager and Actuary, Scottish Equitable Life Assurance Society, and Director, Scottish Industrial Estates. 10 Gillsland Road, Edinburgh 10.	
1933		* Murray, John, M.A. (Aberd.), Ph.D. (Edin.), Rector, The Academy, Annan. 9 Seaforth Avenue, Annan	
1934		* Murray, Walter George Robertson, A.I.C., Technical Assistant, Department of Chemistry, University of Edinburgh (West Mains Road). 20 Montpelier Park, Edinburgh 10	
1935		* Narayana, Basudeva, M.Sc., M.B. (Calcutta), Ph.D. (Edin.), Professor of Physiology, University of Patna. Department of Physiology, Medical College, Patna, India	
1937		* Nasmith, Charles Roy, B.A., M.A. (Honorary), Colgate University, U.S.A., United States Consul in Edinburgh. Abden House, 1 Marchhall Crescent, Edinburgh 9	
1931		Nelson, Alexander, B.Sc. (Glas.), Ph.D. (Edin.), N.D.A., Lecturer in Plant Physiology and Agricultural Botany, University of Edinburgh. 14 Netherby Road, Edinburgh 5	
1924		* Nelson, Philip, M.A., M.D., Ph.D., F.S.A., Beechwood, Calderstones, Liverpool	
1898		Newman, Sir George, G.B.E., K.C.B., M.D., D.C.L., Hon. D.Sc., LL.D., F.R.C.S. (Eng.), F.R.C.P. (Lond.), formerly Chief Medical Officer, Ministry of Health and Board of Education. Grims Wood, Harrow Weald, Middlesex	
1928		* Nichols, James Edward, M.Sc. (Dunelm), Ph.D. (Edin.), Deputy Director, Imperial Bureau of Animal Genetics, University of Edinburgh (West Mains Road)	
1933	C.	* Nicol, Thomas, M.D., Ch.B., D.Sc. (Glas. and Lond.), F.R.C.S.E., Professor of Anatomy and Head of the Department of Anatomy, University of London, King's College, Strand, London, W.C. 2	
1927		* Noble, Thomas Paterson, M.D. (Edin.), F.R.C.S. (Eng.). Hillside, Ebbw Vale, Mon., Wales	

Date of Election			Service on Council, etc.
1934	C.	* Normand, Alexander Robert, M.A., B.Sc., Ph.D. (Edin.), formerly Professor of Chemistry, Wilson College, Bombay. 7 India Street, Edinburgh 3	
1928	C. N.	* O'Donoghue, Charles Henry, D.Sc. (Lond.), Professor of Zoology, University of Reading	Sec. 1936-39.
1925		* Ogg, William Gammie, M.A., Ph.D., Director, Macaulay Institute for Soil Research, Craigiebuckler, Aberdeen; Research Lecturer in Soil Science in the University of Aberdeen	
1923	C.	* Ogilvie, Alan G., O.B.E., M.A., B.Sc. (Oxon.), Professor of Geography, University of Edinburgh (High School Yards). 40 Fountainhall Road, Edinburgh 9	1932-35.
1929		* Ogilvie, Frederick Wolff, M.A. (Oxon.), LL.D., Director-General of the British Broadcasting Corporation. Broadcasting House, London, W. 1	
1939		* Oliphant, William Douglas, B.Sc. Eng. (Edin.), A.M.I.E.E., Scientific Officer to the Air Ministry. St Baldred's Road, North Berwick	
1886		Oliver, James, M.D., F.L.S., Physician to the London Hospital for Women. 123 Harley Street, London, W.	
1895	C.	† Oliver, Sir Thomas, Kt., D.L., M.D., LL.D., F.R.C.P. (Lond.), formerly Vice-Chancellor of the University of Durham. 7 Ellison Place, Newcastle-upon-Tyne	
1930		* Oliver, William, B.Sc., A.M.Inst.C.E., Professor of Organisation of Industry and Commerce, University of Edinburgh (South Bridge). 70 Netherby Road, Trinity, Edinburgh 5	
1939		* Olver, Colonel Sir Arthur, C.B., C.M.G., F.R.C.V.S., Principal, Royal (Dick) Veterinary College, Edinburgh. Prestonfield House, Edinburgh 9	
1930		* O'Riordan, George Francis, B.Sc. Eng., M.I.Mech.E., Principal of Battersea Polytechnic, London, S.W. 7. Hesslewood, 39 Langley Avenue, Surbiton, Surrey	
1924		* Orr, Sir John Boyd, Kt., D.S.O., M.C., M.A., D.Sc., M.D., LL.D., F.R.S., Director of the Rowett Research Institute for Research in Animal Nutrition, Aberdeen	
1915		Orr, Lewis P., F.F.A., formerly General Manager of the Scottish Life Assurance Co., 3 Belgrave Place, Edinburgh 4	
1932		* Orr, Matthew Young, 38 Lennox Row, Edinburgh 5	
1908		Page, William Davidge, formerly of 13 Yarrell Mansions, Queen's Club Gardens, London, W. 14. (*Present address not known*)	
1934		* Pal, Rudrendra Kumar, D.Sc. (Edin.), M.R.C.P.E., F.R.C.S.E., formerly Professor of Physiology, Prince of Wales Medical College, Patna. Nayasarak, Sylhet, Assam, India	
1905		Pallin, Colonel William Alfred, C.B.E., D.S.O., F.R.C.V.S., 5 Tower Gardens, Hythe, Kent	
1933		Parsons, Charles Wynford, M.A., Lecturer in Zoology, University of Glasgow. Mapledene, Ralston Road, Bearsden, Dumbartonshire	
1901		Paterson, David, Leewood, Rosslyn Castle, Midlothian	
1937		* Paterson, Thomas T., B.Sc. (Edin.), Fellow of Trinity College, Cambridge. Swanlea House, Buckhaven, Fife	
1936		* Paterson, William George Rogerson, O.B.E., B.Sc., Principal of the West of Scotland Agricultural College. Buckrigg, Beattock, Dumfriesshire	
1927		* Patterson, Charles, A.M.I.Mar.E., Lecturer in Mechanical Engineering Design, University of Edinburgh (Sanderson Engineering Laboratories, Mayfield Road). 22 Dudley Terrace, Trinity, Edinburgh 6	
1926		* Patton, Donald, M.A., B.Sc., Ph.D., Lecturer in Science, Glasgow Provincial College for the Training of Teachers. 15 Jordanhill Drive, Glasgow, W. 3	
1923	C.	* Peacock, Alexander David, D.Sc., Professor of Zoology, University College, Dundee	1935-38.
1914		Pearson, Joseph, D.Sc., F.L.S., formerly Director of the Colombo Museum, and Marine Biologist to the Ceylon Government. Director of the Tasmanian Museum, Hobart, Tasmania	
1904		Peck, Sir James Wallace, Kt., C.B., M.A., Secretary to the Scottish Education Department, Chief Divisional Food Officer for Scotland, St Andrew's House, Edinburgh	1926-28.
1887	C. M-B.	Peddie, William, D.Sc., Professor of Natural Philosophy, University College, Dundee. The Weisha, Ninewells, Dundee	1904-07, 1908-11, 1933-36. V-P 1919-22.

Date of Election			Service on Council, etc.
1925		* Penman, David, C.I.E., D.Sc., M.Inst.M.E., Chief Inspector of Mines in India. St Agnes, Viewforth Street, Kirkcaldy	
1931	C.	* Phemister, James, M.A., D.Sc. (Glas.), Petrographer, H.M. Geological Survey and Museum, Exhibition Road, London, S.W. 7	
1938		* Phemister, Thomas Crawford, D.Sc. (Glas.), Ph.D. (Cantab.), F.G.S., Professor of Geology, University of Aberdeen	
1907	C.	† Phillips, Charles E. S., O.B.E., Castle House, Shooters Hill, Woolwich, S.E. 18	
1929	C.	* Phillips, John Frederick Vicars, D.Sc., F.L.S., Professor of Botany, University of the Witwatersrand, Johannesburg, Union of South Africa	
1935		* Pichamuthu, Charles Solomon, B.Sc. (Mysore), Ph.D. (Glas.), F.G.S., Assistant Professor of Geology, University of Mysore. Central College, Bangalore, India	
1932		* Pickard, James Nichol, B.A. (Cantab.), Ph.D. (Edin.), Lynburn, Carlops, Penicuik, Midlothian	
1928		* Pilcher, Robert Stuart, General Manager, Manchester Corporation Tramways. 55 Piccadilly, Manchester	
1908	C.	† Pirie, James Hunter Harvey, B.Sc., M.D., F.R.C.P.E., Research Pathologist and Bacteriologist, South African Institute for Medical Research. P.O. Box 1038, Johannesburg, South Africa	
1911		Pirie, James Simpson, M.Inst.C.E., 25 Grange Road, Edinburgh 9	
1906		Pitchford, Herbert Watkins, C.M.G., F.R.C.V.S., Victoria Club, Pietermaritzburg, South Africa	
1934		* Plenderleith, Harold James, M.C., B.Sc., Ph.D. (St Andrews), F.C.S., Deputy Keeper in Charge, Research Laboratory of British Museum, London. 198 Willesden Lane, London, N.W. 6	
1937		* Pollock, Sir John Donald, Bart., O.B.E. (Mil.), D.L., M.D., Hon. D.Sc. (Oxon.), LL.D. (Edin.), Rector of the University of Edinburgh. Manor House, Boswall Road, Edinburgh 5	
1919		* Porritt, B. D., M.Sc. (Lond.), F.I.C., F.Inst.P., Director of Research, Research Association of British Rubber Manufacturers, 105–7 Lansdowne Road, Croydon, Surrey. 23 Addiscombe Grove, Croydon, Surrey	
1888		† Prain, Sir David, Kt., C.M.G., C.I.E., M.A., M.B., LL.D., F.R.S., F.L.S., Lt.-Col., I.M.S. (retired), formerly Director, Royal Botanic Gardens, Kew, Surrey. The Well Farm, Whyteleafe, Surrey	
1937		* Prasad, Badri Narayan, Ph.D. (Edin.), M.Sc., M.B., D.T.M. (Cal.), Lecturer in Pharmacology, P. W. Medical College, Bankipore, P.O., Bihar, India	
1932		* Prasad, Gorakh, D.Sc. (Edin.), Reader in Mathematics, University of Allahabad. Beli Road, Allahabad, India	
1926	C.	* Prashad, Baini, D.Sc., Superintendent, Zoological Survey of India, Indian Museum, Calcutta	
1933		* Preston, Frank Anderson Baillie, L.R.I.B.A., F.S.A.Scot., Lecturer in Municipal Engineering, Royal Technical College, Glasgow. Craigrownie, Briarwell Road, Milngavie	
1915		† Price, Frederick William, M.D. (Edin.), C.M., F.R.C.P. (Lond.), Consulting Physician to the Royal Northern Hospital, London; Senior Physician to the National Hospital for Diseases of the Heart. 133 Harley Street, London, W.	
1932		* Price, Thomas Slater, O.B.E., D.Sc. (Lond., Birm.), Ph.D. (Leip.), F.R.S., Professor of Chemistry, Heriot-Watt College, Edinburgh. 2 Cluny Drive, Edinburgh 10	
1932		* Pringle, John, Hon. D.Sc., F.G.S., formerly Palæontologist, Geological Survey of Great Britain. 28 Bowen Road, Harrow, Middlesex	
1920	C.	* Purser, George Leslie, M.A. (Cantab.), F.Z.S., Lecturer in Embryology, University of Aberdeen	
1898		Purves, John Archibald, D.Sc., Chilliswood, Trull, Taunton	
1936	C.	* Raitt, Douglas Stewart, D.Sc., Ph.D. (Aberd.), F.L.S., Naturalist, Fishery Board for Scotland, Aberdeen. 41 Rosehill Drive, Aberdeen	
1899	C.	Ramage, Alexander G., Lochcote, Linlithgowshire	
1938		* Ramsay, Andrew Maitland, M.D., LL.D., F.R.F.P.S.G., formerly Lecturer in Ophthalmology, University of Glasgow. The Castle House, St Andrews	
1904		Ratcliffe, Joseph Riley, M.B., C.M., c/o The Librarian, University, Birmingham	
1900		Raw, Nathan, C.M.G., M.D., 22 Ashworth Road, Maida Vale, London, W. 9	

Date of Election			Service on Council, etc.
1937		Rawlins, Francis Ian Gregory, M.Sc. (Cantab.), F.Inst.P., Scientific Adviser to the Trustees of the National Gallery, London. 5 The Mount Square, Hampstead, London, N.W. 3	
1927	C.	* Read, Herbert Harold, D.Sc. (Lond.), A.R.C.S., F.R.S., F.G.S., formerly George Herdman Professor of Geology, University of Liverpool; Professor of Geology in the University of London Imperial College of Science and Technology, London, S.W. 7	
1929		* Read, Selwyn, B.A., Schoolmaster, Edinburgh Academy. Mackenzie House, Kinnear Road, Edinburgh 4	
1902		Rees-Roberts, John Vernon, M.D., D.Sc., D.P.H., 90 Fitzjohns Avenue, Hampstead, London, N.W. 3	
1913		Reid, Harry Avery, O.B.E., F.R.C.V.S., D.V.H., Veterinary Officer to the New Zealand Government. C/o High Commissioner for New Zealand, 415 Strand, London, W.C. 2	
1936		* Renouf, Louis Percy Watt, B.A. (Cantab.), M.Sc. (Nat. Univ. Ireland), Professor of Zoology, University College, Cork, Director of the University of Cork Biological Station. St Philomena's, Tivoli, Cork, I.F.S.	
1914		Renshaw, Graham, M.D., M.R.C.S. (Eng.), L.R.C.P. (Lond.), L.S.A., F.Z.S., Lecturer in Zoology, Extramural Department, University of Manchester. Editor of *Natureland*, Sale Bridge House, Sale, Manchester	
1913		† Richardson, Harry, O.B.E., M.C., M.Inst.E.E., M.Inst.M.E., 72 Oakwood Court, London, W. 14	
1908		Richardson, Linsdall, P.A.Inst.W.E., F.G.S. 104 Greenfield Road, Harbourne, Birmingham	
1927		* Richey, James Ernest, M.C., B.A., B.A.I. (T.C.D.), Sc.D., F.R.S., F.G.S., District Geologist, H.M. Geological Survey (Scotland), 19 Grange Terrace, Edinburgh 9	1938–
1930		* Ritchie, Allan Watt, M.B.E., F.R.San.I., Chief Sanitary Inspector, City of Edinburgh, 8 Cumlodden Avenue, Edinburgh 12	
1916	C.	* Ritchie, James, M.A., D.Sc., Professor of Natural History, University of Edinburgh. 31 Mortonhall Road, Edinburgh 9	1921–24, 1926–28, 1937– Sec. 1928–31. V-P 1931–34.
1914	C.	Ritchie, James Bonnyman, D.Sc., Rector, The Academy, Ayr. 35 Carrick Road, Ayr	
1937		* Ritchie, Mowbray, Ph.D., D.Sc. (Edin.), Lecturer in Chemistry, University of Edinburgh (West Mains Road)	
1906	C.	Ritchie, William Thomas, O.B.E., M.D., F.R.C.P.E., formerly Professor of Medicine, University of Edinburgh (Teviot Place). 22 Alva Street, Edinburgh 2	
1919		* Ritchie-Scott, Alexander, B.Sc. (Edin.), D.Sc. (Lond.), 72 Station Road, Barnes Green, London, S.W. 13	
1936		* Robb, James, M.A., B.D., LL.B., LL.D. (St Andrews), Secretary to the Carnegie Trust for the Universities of Scotland. 26 Ormidale Terrace, Edinburgh 12	
1929	C.	* Robb, Richard Alexander, M.A., D.Sc., Lecturer in Mathematics, University of Glasgow. 27 Moor Road, Eaglesham, Renfrewshire	
1931		* Robb, William, N.D.A., Director of Research, Scottish Society for Research in Plant Breeding. Craigs House, Corstorphine, Edinburgh 12	
1919		* Roberts, Alfred Henry, O.B.E., M.Inst.C.E., formerly Superintendent and Engineer, Leith Docks. Dohnavur, Ravelston Dykes, Edinburgh 4	
1926		* Roberts, John Alexander Fraser, M.A. (Cantab.), M.B., D.Sc., Stoke Park Colony, Stapleton, Bristol	
1937		* Robertson, John Watson, M.A., B.Sc. (Aberd.), Headmaster, Central School, Aberdeen. 21 Belvedere Crescent, Aberdeen	
1937		* Robertson, Thomas Graham, The Hon. Lord Robertson, Senator of the College of Justice. 14 India Street, Edinburgh 3	1938–
1919		* Robertson, William Alexander, F.F.A., Century Insurance Co., Ltd., 18 Charlotte Square. Mardale, 3 Buckstone Park, Edinburgh 10	
1896	C.	Robertson, W. G. Aitchison, D.Sc., D.Litt., M.D., F.R.C.P.E., Barrister-at-Law, Lincoln's Inn. St Margarets, St Valerie Road, Bournemouth	

Date of Election			Service on Council, etc.
1932	C.	* Robson, John Michael, M.D., B.Sc., Lecturer in Materia Medica, University of Edinburgh (Teviot Place). 9 Golf Course Road, Bonnyrigg, Midlothian	
1926		* Romanis, William Hugh Cowie, M.A., M.B., M.C. (Cantab.), F.R.C.S. (Eng.), Surgeon to St Thomas's Hospital, London. 120 Harley Street, London, W. 1	
1916		* Ronald, David, M.Inst.C.E., formerly Chief Engineer, Scottish Department of Health. 72A George Street, Edinburgh 2	
1938		* Rosebery, The Right Hon. The Earl of, D.S.O., M.C., Lord Lieutenant of Midlothian. Dalmeny House, Edinburgh	
1909	C.	Ross, Alexander David, M.A., D.Sc., F.Inst.P., F.R.A.S., Professor of Physics, University of Western Australia, Perth, Western Australia	
1921		* Ross, Edward Burns, M.A., formerly Professor of Mathematics, Madras Christian College, Madras. 41 Liberton Brae, Edinburgh 9	
1935		* Rowatt, Thomas, O.B.E., M.I.Mech.E., F.S.A.Scot., Director, Royal Scottish Museum. Spottiswoode, Spylaw Bank Road, Colinton, Edinburgh 13	
1931	C. K.	* Ruse, Harold Stanley, M.A. (Oxon.), D.Sc., Professor of Mathematics, University College, Southampton	
1906		Russell, Alexander Durie, B.Sc., F.R.A.S., formerly Head Mathematical Master, Falkirk High School. 12 Heugh Street, Falkirk	
1930		Russell, David, LL.D., Paper Manufacturer. Silverburn, Leven, Fife	
1902	C. K.	Russell, James, 22 Glenorchy Terrace, Edinburgh 9	
1937		* Russell, William Ritchie, M.D. (Edin.), F.R.C.P.E., M.R.C.P. (Lond.), Assistant Physician to the Royal Infirmary and to the Deaconess Hospital, Edinburgh. 8 Randolph Cliff, Edinburgh 3	
1934		* Rutherford, Daniel Edwin, M.A., B.Sc. (St Andrews), D.Math. (Amsterdam), Lecturer in Applied Mathematics, United College, University of St Andrews. 5 John Street, St Andrews	
1925	C.	* Saddler, William, M.A., B.A., Professor of Mathematics, Canterbury College, Christchurch, N.Z.	
1906		Saleeby, Caleb Williams, M.D., 13 Greville Place, Hampstead, London, N.W. 6	
1916	C.	* Salvesen, The Rt. Hon. Lord, P.C., K.C., LL.D., Judge of the Court of Session (retired), Dean Park House, Edinburgh 4	1920–22. V.P 1922–25.
1934		* Salvesen, Harold Keith, M.A. (Oxon. and Harvard), Captain (retired, I.A.), Shipowner, Inveralmond, Cramond, Midlothian	
1914		Salvesen, Colonel Theodore Emile, of Culrain, F.R.S.A., F.S.A.Scot., Chevalier de la Légion d'Honneur. 37 Inverleith Place, Edinburgh 4, and Carbisdale Castle, Ardgay, Ross-shire	
1912	C. K.	Sampson, Ralph Allen, M.A., D.Sc., LL.D., F.R.S., formerly Astronomer Royal for Scotland and Professor of Astronomy, University of Edinburgh. Greenhill, 20 Observatory Road, Edinburgh 9. (*Died 7th November* 1939)	1912–15, 1919–21. V.P 1915–18, 1933–36. Sec. 1922–23. Gen. Sec. 1923–33.
1927	C.	* Sandeman, Ian, M.A., D.Sc., Ph.D. (St Andrews), Acting Chief Inspector of Schools, Education Department, Colombo, Ceylon	
1938		* Sandilands, James, A.H.-W.C., F.I.C., Senior Lecturer in Chemistry, Heriot-Watt College. 102 Westholmes Gardens, Musselburgh	
1930		* Sansome, Frederick Whalley, Ph.D., F.L.S., Senior Lecturer in Horticulture, Botany Department, University of Manchester	
1922		* Sarkar, Bijali Behari, M.Sc., D.Sc. (Edin.), Lecturer in Physiology, University, Calcutta. 33/3 Lansdowne Road, Calcutta	
1903		Sarolea, Charles, Ph.D., D.Litt., LL.D. (Montreal), formerly Professor of French, University of Edinburgh. 21 Royal Terrace, Edinburgh 7	
1935		* Say, Maurice George, Ph.D., M.Sc. (Lond.), M.I.E.E., Professor of Electrical Engineering, Heriot-Watt College, Edinburgh. Dreghorn Loan, Colinton, Edinburgh	
1927	C.	* Schlapp, Robert, M.A. (Edin.), Ph.D. (Cantab.), Lecturer in Applied Mathematics, University of Edinburgh (Drummond Street). 40A Morningside Park, Edinburgh 10	
1885	C.	† Scott, Alexander, M.A., D.Sc., F.R.S., formerly Director of Scientific Research at the British Museum. 117 Hamilton Terrace, London, N.W. 8	

Date of Election			Service on Council, etc.
1919		* Scott, Alexander, M.A., D.Sc., 3 Winton Terrace, Stoke-on-Trent	
1917		* Scott, Henry Harold, C.M.G., M.D., F.R.C.P. (Lond.), M.R.C.S. (Eng.), D.P.H., Director, Bureau of Hygiene and Tropical Diseases, Keppel Street, Gower Street, London, W.C. 1	
1928		* Senior-White, Ronald, F.R.E.S., Malariologist, Bengal-Nagpur Railway, Kidderpore, P.O., Calcutta, India	
1930		* Shankland, Ernest Claud, F.R.Met.S., River Superintendent, Port of London Authority. Mariners, Balfour Gardens, Folkestone	
1927		* Sharpley, Forbes Wilmot, B.Sc. Eng. (Lond.), Ph.D., M.Inst.E.E., Professor of Electrical and Mechanical Engineering, Indian School of Mines, Dhanbad, India	
1931		* Shaw, John James M'Intosh, M.A., M.D., F.R.C.S.E., Lecturer in Surgery and Clinical Surgery, University of Edinburgh. Greenaway, Kinnear Road, Edinburgh 4	
1927		* Shearer, Ernest, M.A., B.Sc. (Edin.), Professor of Agriculture and Rural Economy, University of Edinburgh, and Principal, Edinburgh and East of Scotland College of Agriculture, 13 George Square, Edinburgh 8	
1931		* Shearer, James Fleming, M.A., B.Sc., Ph.D., Lecturer in Natural Philosophy, University of Glasgow. 149 Queen's Drive, Wavertree, Liverpool 15	
1932		* Simpson, Alexander Rudolf Barbour, B.Sc. (Edin.), M.A. (Cantab.), F.R.G.S., Hillstone School, Malvern, Worcestershire	
1908		Simpson, George Freeland Barbour, J.P., M.D., F.R.C.P.E., F.R.C.S.E., 43 Manor Place, Edinburgh 3	
1932	C.	* Simpson, John Baird, D.Sc. (Aberd.), Senior Geologist, H.M. Geological Survey (Scotland), 19 Grange Terrace, Edinburgh 9	
1938		* Sinclair, Arthur Henry Havens, M.D., F.R.C.S.E., Hon. Surgeon-Oculist to H.M. The King in Scotland, Consulting Ophthalmic Surgeon, Royal Infirmary, Edinburgh. 6 Charlotte Square, Edinburgh 2	
1939		* Sinclair, William Angus, M.A. (Edin.), Lecturer in Philosophy, University of Edinburgh (South Bridge). 5 Great Stuart Street, Edinburgh 3	
1900		Sinhjee, Sir Bhagvat, G.C.I.E., M.D., LL.D. (Edin.), H.H. the Thakur Sahib of Gondal, Kathiawar, Bombay, India	
1903		† Skinner, Robert Taylor, J.P., M.A., F.S.A.Scot., formerly House-Governor, Donaldson's Hospital. 35 Campbell Road, Edinburgh 12	
1937		* Slater, James Kirkwood, M.B. (Edin.), F.R.C.P.E., Assistant Physician, Royal Infirmary, Edinburgh, and Physician to the Deaconess Hospital, Edinburgh. 7 Walker Street, Edinburgh 3	
1930	C.	* Slater, Robert Henry, D.Sc., Ph.D. (Edin.), F.I.C., Department of Chemical Pathology, St Mary's Hospital, London, W. 2	
1929		* Smail, James Cameron, O.B.E., Companion Inst.E.E. (VICE-PRESIDENT), Principal, Heriot-Watt College, Edinburgh. 1 Grange Terrace, Edinburgh 9	1934-37. V-P 1937-
1926	C.	* Small, James, D.Sc., Professor of Botany, Queen's University, Belfast. Orkla, 50 Myrtlefield Park, Belfast	
1901		Smart, Edward, B.A., B.Sc., Tillyloss, Tullylumb Terrace, Perth. (*Died 5th December* 1939)	
1920		* Smellie, William Robert, M.A., D.Sc., Geologist on the Staff of the Anglo-Persian Oil Company. Baron Cliff, Cove, Dumbartonshire	
1937		* Smith, Alexander Martin, Ph.D., D.Sc. (Edin.), A.I.C., Lecturer in Agricultural Chemistry, University of Edinburgh, 13 George Square. 1 Mortonhall Road, Edinburgh 9	
1928		Smith, Alick Drummond Buchanan, O.B.E. (Mil.), M.A., B.Sc. (Agric.) (Aberd.), D.Sc. (Edin.), M.S.A. (Iowa), Lecturer, Institute of Animal Genetics, University of Edinburgh (West Mains Road)	
1937		* Smith, Horace George, B.Sc. (Glas.), Ph.D. (Bristol), Assistant, Department of Natural History, Marischal College, University of Aberdeen	
1921		* Smith, Norman Kemp, M.A., D.Phil., D.Litt., LL.D., Professor of Logic and Metaphysics, University of Edinburgh (South Bridge). 14 Kilgraston Road, Edinburgh 9	
1923		* Smith, Percy James Lancelot, M.A. (Oxon.), F.I.C., F.C.S., Science Master, Loretto School. 47 Dalrymple Loan, Musselburgh	
1911		Smith, Stephen, B.Sc., 34 Craigmillar Park, Edinburgh 9	
1929		* Smith, Sydney Alfred, M.D., F.R.C.P.E., D.P.H., Professor of Forensic Medicine, University of Edinburgh (Teviot Place). 10 Oswald Road, Edinburgh 9	1936-39.

Date of Election			Service on Council, etc.
1919		* Smith, Sir William Wright, Kt., M.A., D. ès Sc., Regius Professor of Botany, University of Edinburgh, Regius Keeper of the Royal Botanic Garden, and King's Botanist in Scotland. Inverleith House, Edinburgh 4	Sec. 1923–28. V.P 1928–31.
1932		* Sneeden, Jean-Baptiste Octave, B.Sc., Ph.D. (Glas.), Lecturer on Heat Engines, Royal Technical College, Glasgow. 39 Kingshouse Avenue, Glasgow, S. 4	
1899		Snell, Ernest Hugh, M.D., B.Sc., D.P.H. (Cantab.), Barrister-at-Law, formerly Medical Officer of Health, Coventry. 3 Eaton Road, Coventry	
1933		* Somerville, John Livingston, C.A., Auditor, University of Edinburgh. 8 Ravelston Park. Edinburgh 4	
1929		* Southwell, Thomas, D.Sc., A.R.C.S., formerly Walter Myers Lecturer in Parasitology, School of Tropical Medicine, University of Liverpool. Stansfield Cottage, Todmorden, Lancashire	
1925		* Staig, Robert Arnot, M.A., Ph.D., Lecturer in Zoology, University of Glasgow. Glenlea, Lasswade, Midlothian	
1891		Stanfield, Richard, A.R.S.M., M.Inst.C.E., Emeritus Professor of Mechanics and Engineering, Heriot-Watt College, Edinburgh. 24 Mayfield Gardens, Edinburgh 9	1926–29.
1923		* Stebbing, Edward Percy, M.A., Professor of Forestry, University of Edinburgh (George Square)	
1923		* Stenhouse, Andrew G., F.G.S., 191 Newhaven Road, Edinburgh 6	
1929	C.	* Stephen, Alexander Charles, D.Sc., Keeper, Natural History Department, Royal Scottish Museum, Edinburgh. Eastcroft, Cramond Bridge, Edinburgh 4	
1931		* Steven, George Alexander, B.Sc. (Edin.), Assistant Naturalist, Marine Laboratory, Plymouth. 1 Seaview Villas, Pentyre Terrace, Plymouth, Devon	
1937	C.	* Steven, Henry Marshall, B.Sc., Ph.D. (Edin.), M.A., Hon. Causa (Oxon.), Professor of Forestry, University of Aberdeen. 75 Fountainhall Road, Aberdeen	
1886	C.	Stevenson, Charles A., B.Sc., M.Inst.C.E., Radella, North Berwick	
1919		* Stevenson, David Alan, B.Sc., M.Inst.C.E., 22 Glencairn Crescent, Edinburgh 12	
1935		* Stevenson, Eric, B.Sc. (Edin.), A.M.I.Mech.E., Lecturer in Engineering, University of Edinburgh (Sanderson Engineering Laboratory, Mayfield Road). 7 Beauchamp Road, Edinburgh 9	
1936		* Stewart, Alexander Dron, C.I.E., M.B., Ch.B. (Edin.), F.R.C.S.E., Lt.-Col. I.M.S. (retired), Superintendent of the Royal Infirmary, Edinburgh. Meadow Walk House, Edinburgh 3	
1939		* Stewart, Bernard Halley, M.A., M.D., B.Ch. (Cantab.), President of the Sir Halley Stewart Trust. Fairlight, Totteridge, London, N. 20	
1925		* Stewart, David Smith, Ph.D., M.Inst.C.E., Lecturer on Structural Engineering Drawing, University of Edinburgh (Sanderson Engineering Laboratory, Mayfield Road). 82 Lasswade Road, Edinburgh 9	
1938		* Stewart, James, M.A., D.Sc., Ph.D., Chief Biochemist, Moredun Institute, Animal Diseases Research Association, Gilmerton. 33 Hatton Place, Edinburgh 9	
1938		* Stewart, John Livingstone, B.Sc. (Edin.), M.R.C.V.S., Director of Veterinary Services, Gold Coast, Department of Animal Health, Pong-Tamale. P.O. Box No. 32, Tamale, Northern Territories, Gold Coast	
1924		* Stiles, Sir Harold Jalland, K.B.E., Kt., M.B., F.R.C.S.E., D.Sc. (Hon.), LL.D., Emeritus Professor of Clinical Surgery, University of Edinburgh. Whatton Lodge, Gullane, E. Lothian	1934–37.
1902		Stockdale, Herbert Fitton, LL.D., formerly Director of the Royal Technical College, Glasgow. Clairinch, Upper Helensburgh, Dumbartonshire	
1889	C.	Stockman, Ralph, M.D., LL.D., F.R.C.P.E., F.R.F.P.S.G. (VICE-PRESIDENT), Emeritus Professor of Materia Medica and Therapeutics, University of Glasgow. White Lodge, Barnton Avenue, Edinburgh 4	1903–05, 1936–39. V.P. 1939–
1926		* Stokoe, William Norman, B.Sc., Ph.D. (Lond.), Chief Chemist, Craigmillar Creamery Co., Ltd. 8 Cobden Road, Edinburgh 9	
1906		Story, Fraser, O.B.E., formerly Professor of Forestry, University College, Bangor, North Wales. The Wall House, Yorke Road, Reigate, Surrey	
1907		Strong, John, C.B.E., M.A., LL.D., Emeritus Professor of Education, University of Leeds. C/o The Librarian, The University, Leeds	

Date of Election			Service on Council, etc.
1930	C.	* Stump, Claude Witherington, M.D., D.Sc., Professor of Embryology and Histology, University of Sydney	
1937		* Suffolk and Berkshire, The Rt. Hon. The Earl of, Charlton Park, Malmesbury, Wilts	
1935		* Sutherland, John Derg, B.Sc., B.Ed., Ph.D. (Edin.), M.B., Ch.B., Lecturer in Psychology, University of Edinburgh. 2 Windsor Street, Edinburgh 7	
1930		* Sutherland, Sir John Donald, C.B.E., LL.D., Legion of Honour (France), Order of Leopold (Belgium), formerly Forestry Commissioner, Scotland. 11 Inverleith Row, Edinburgh 4	
1925		Sutton, Richard L., M.D., D.Sc., LL.D., 1308 Bryant Building, 1102 Grand Avenue, Kansas City, Mo., U.S.A.	
1932		* Swinton, William Elgin, B.Sc., Ph.D. (Glas.), F.L.S., F.Z.S., F.G.S., Curator of Fossil Reptiles and Amphibia, British Museum (Natural History), South Kensington, London, S.W. 7	
1933		* Tait, John Barclay, B.Sc. (Edin.), Ph.D., A.H.-W.C., Senior Naturalist (Hydrographer), Marine Laboratory (Fisheries Division, Scottish Home Department), Aberdeen. 23 Cromwell Road, Aberdeen	
1937		* Tait, John Guthrie, Scholar of Peterhouse College, Cambridge, B.A. (Cantab.), Barrister-at-Law, Lincoln's Inn, Hon. Fellow, Madras University, formerly Principal, Central College, Bangalore, India. 38 George Square, Edinburgh 8	
1890	C.	† Tanakadate, Aikitu, Hon. Professor of Natural Philosophy, Imperial University of Japan. Koisikawa, Zôsigayamati, 144, Tokyo, Japan	
1933		* Taylor, George, D.Sc. (Edin.), F.L.S., Assistant Keeper, Department of Botany, British Museum. Ballochmyle, Loudwater, Rickmansworth, Herts	
1899		Taylor, James, M.A., formerly Mathematical Master, Edinburgh Academy. 18 Hillview, Blackhall, Edinburgh 4	
1885	C.	Thompson, Sir D'Arcy Wentworth, Kt., C.B., M.A., LL.D. (Aberd., Edin.), Hon. D.Sc. (Dublin, Witwatersrand), D.Litt., F.R.S., Professor of Natural History, University, St Andrews. 44 South Street, St Andrews	1892–95, 1896–99, 1907–10, 1912–15, 1922–25. V-P 1916–19. Curator 1926–34. P 1934–39.
1932		Thompson, Harold William, D.Litt. (Edin.), A.M., Ph.D. (Harvard), F.S.A. Scot., Professor of English, N.Y. State College, Albany, N.Y., U.S.A.	
1917	C. N.	* Thompson, John M'Lean, M.A., D.Sc., F.L.S., Professor of Botany, University of Liverpool	
1938		* Thomson, Arthur Landsborough, C.B., O.B.E., M.A., D.Sc. (Aberd.), Principal Assistant Secretary, Medical Research Council. 16 Tregunter Road, London, S.W. 10	
1896		Thomson, George Ritchie, C.M.G., M.B., C.M., formerly Professor of Surgery, University of the Witwatersrand, Johannesburg, Transvaal. Hordle Grange, Hordle, Hants	
1903		Thomson, George S., Kenmore Farm, Whelpley Hill, Chesham, Bucks	
1906		Thomson, Gilbert, M.A., M.Inst.C.E., 164 Bath Street, Glasgow, C. 2	
1926		* Thomson, Godfrey Hilton, D.Sc., Ph.D., Professor of Education, University of Edinburgh (Moray House)	1931–34.
1926	C.	* Thomson, John, M.A., Ph.D. (Glas.), Lecturer in Plant Physiology, University of Glasgow. 2 Chartwell Terrace, Bearsden, Glasgow	
1934		* Thomson, Matthew Sydney, M.A., M.D., B.Ch. (Cantab.), F.R.C.P. (Lond.), M.R.C.S. (Eng.), Physician for Diseases of the Skin, King's College Hospital, Belgrave Hospital for Children. 106 Harley Street, London, W. 1	
1912		Thomson, R. Tatlock, F.I.C., 156 Bath Street, Glasgow	
1938		* Thomson, Robert, B.Sc. Mech.Eng., A.M.I.Mech.E., Ph.D. (Edin.), Education Officer, Air Ministry. Moorfield, Terrick Road, Butler's Cross, Aylesbury, Bucks	
1882		Thomson, Sir William, Kt., M.A., B.Sc., LL.D., formerly Principal, University of the Witwatersrand. Dunedin, Glencairn, Simonstown, South Africa	
1917		* Thorneycroft, Wallace, J.P., Chalmington, Dorchester	

Date of Election			Service on Council, etc.
1933		* Timms, Geoffrey, Ph.D. (Cantab.), Lecturer in Mathematics, University of St Andrews. 8 St Mary's Street, St Andrews	
1937		* Tod, Henry, B.Sc., Ph.D. (Edin.), Biochemist, The Royal Edinburgh Hospital for Mental Disorders. 35 Oxgangs Road, Edinburgh 10	
1920		* Todd, John Barber, B.Sc., Ph.D., M.I.Mech.E., Reader in Engineering, University of Edinburgh. 4 Bright's Crescent, Edinburgh 9	
1938		* Topping, Andrew, T.D., M.A., M.D., D.P.H. (Aberd.), A Senior Medical Officer, Public Health Department, London County Council, County Hall, London, S.E. 1	
1917		* Tovey, Sir Donald Francis, Kt., B.A. (Oxon.), M.Mus. (Hon.), Birmingham, Professor of Music, University of Edinburgh (Reid School of Music). 39 Royal Terrace, Edinburgh 7	
1914		† Tredgold, Alfred Frank, M.D. (Durham), F.R.C.P. (Lond.), Lecturer on Mental Deficiency, University of London. St Martins, Guildford	
1915		Trotter, George Clark, M.D. (Edin.), D.P.H. (Aberd.), F.S.A.Scot., Medical Officer of Health, Metropolitan Borough, Islington. Braemar, 17 Haslemere Road, Crouch End, London, N. 8	
1938		* Trueman, Arthur Elijah, D.Sc., F.G.S., Professor of Geology, University of Glasgow. 20 Queensborough Gardens, Glasgow	
1922	C. K.	* Turnbull, Herbert Westren, M.A., F.R.S., Professor of Mathematics, University of St Andrews. Randa, Hepburn Gardens, St Andrews	1928–31.
1937		* Turnbull, Mathew McKerrow, M.A. (Edin.), Lecturer in Banking, University of Edinburgh (South Bridge). 14 Minto Street, Edinburgh 9	
1925		* Turner, Harry Moreton Stanley, M.B.E., M.D., M.R.C.S. (Eng.), L.R.C.P. (Lond.), D.T.M. and H., Chevalier de l'Ordre Royale du Sauveur de Grèce, Wing Commander, R.A.F. (retired). The Haven, Brookwood, Surrey	
1924		* Turner, Richard, O.B.E., M.B., C.M. *Temporary Address*, 11 Merchiston Avenue, Edinburgh 10	
1918	C. N	* Tyrrell, George Walter, A.R.C.S., D.Sc., F.G.S., Lecturer in Petrology, Geological Department, University of Glasgow	1926–29. 1937–
1930	C.	* Voge, Cecil Innes Bothwell, Ph.D. (Edin.), 46 Roxborough Park, Harrow-on-the-Hill, London	
1932		* Wade, Henry, C.M.G., D.S.O., M.D., Senior Lecturer in Clinical Surgery, University of Edinburgh (Royal Infirmary). 6 Manor Place, Edinburgh 3	
1926		* Wakeley, Cecil Pembrey Grey, F.R.C.S. (Eng.), L.R.C.P. (Lond.), Surgeon to King's College Hospital, London, Lecturer in Anatomy, King's College, London. 14 Devonshire Street, Portland Place, London, W. 1	
1925	C.	* Walker, Frederick, M.A., Ph.D., D.Sc., Professor of Mineralogy and Geology, The University, Cape Town	
1938		* Walker, Oswald James, B.Sc., Ph.D. (Edin.), Lecturer in Chemistry, University College, Gower Street, London, W.C. 1. 41 Maresfield Gardens, London, N.W. 3	
1931		* Walker, William James Stirling, Ph.D. (Edin.), A.H.-W.C., F.I.C., F.I.L., F.S.A. Scot. 13, The Lawns, Lee Terrace, Blackheath, London, S.E. 3	
1898		Wallace, William, M.A., Campsie, Alta, Canada	
1920		* Walmsley, Thomas, M.D. (Glas.), Professor of Anatomy, Queen's University, Belfast	
1931	C.	* Walton, John, M.A. (Cantab.), D.Sc. (Manch.) (VICE-PRESIDENT), Regius Professor of Botany, University of Glasgow. 23 Lilybank Gardens, Glasgow, W. 2	1934–37. V-P 1937–
1927	C.	* Wardlaw, Claude Wilson, D.Sc. (Glas.), Imperial College of Tropical Agriculture, Trinidad, B.W.I.	
1936		* Warr, The Very Rev. Charles Laing, C.V.O., M.A., D.D., LL.D., Hon. R.S.A., Dean of the Thistle and of the Chapel Royal in Scotland, Chaplain to H.M. The King, Minister of St Giles' Cathedral, Edinburgh. 63 Northumberland Street, Edinburgh 3	
1923		* Warren, John Alexander, M.Inst.C.E. 74 Balshagray Avenue, Partick, Glasgow	
1901	C.	Waterston, David, M.A., M.D., F.R.C.S.E., Professor of Anatomy, University of St Andrews	1916–19, 1925–28.
1927		* Watson, Charles Brodie Boog, F.S.A.Scot. 24 Garscube Terrace, Edinburgh 12	
1923		* Watson, H. Ferguson, M.D., F.R.F.P.S., Ph.D., D.P.H. (Glas.), formerly H.M. Senior Deputy Commissioner, General Board of Control for Scotland. 109 Montgomery Street, Edinburgh 7	

Date of Election			Service on Council, etc.
1923	C.	* Watson, William, M.A., B.Sc. (Edin.), Lecturer in Physics, Heriot-Watt College, Edinburgh. 17 Braidburn Crescent, Edinburgh 10	
1911		† Watt, James, W.S., F.F.A., LL.D. (Vice-President), 7 Blackford Road, Edinburgh 9	1924–26. Treasurer 1926–37. V-P 1937–
1933		* Watt, John Mitchell, M.B., Ch.B., M.R.C.P.E., F.R.S.S.A., Professor of Pharmacology, University of Witwatersrand, Johannesburg, South Africa	
1911		Watt, Very Rev. Lauchlan MacLean, M.A., D.D., LL.D., Kinloch, Lochcarron, Ross-shire	
1928		* Watters, Alexander Marshall, M.A., B.Sc. (Glas.), Rector of Hawick High School. High School House, Hawick	
1896		† Webster, John Clarence, C.M.G., M.D., D.Sc., LL.D., F.R.C.P.E., formerly Professor of Obstetrics and Gynæcology, Rush Medical College, Shediac, N.B., Canada	
1907	C. M-B.	† Wedderburn, Ernest Maclagan, O.B.E., M.A., D.Sc., LL.D., Deputy Keeper of the Signet (Treasurer). 6 Succoth Gardens, Edinburgh 12	1913–16, 1921–24, 1932–35, 1936–37. Treasurer 1937–
1903	C. M-B.	† Wedderburn, J. H. Maclagan, M.A., D.Sc., F.R.S., Professor of Mathematics, Princeton University. Fine Hall, Princeton, N.J., U.S.A.	
1904		Wedderspoon, William Gibson, M.A., LL.D.	
1934	C.	* Weir, John, M.A., Ph.D., D.Sc. (Glas.), Lecturer in Palæontology, University of Glasgow. 18 Botanic Crescent, Glasgow, N.W.	
1930		* White, Adam Cairns, M.B., Ch.B., Ph.D., Assistant Pharmacologist, Wellcome Physiological Research Laboratory, Beckenham, Kent	
1933		* Whitley, William Frederic James, M.D. (Edin.), D.P.H. (Oxon.), Medical Officer of Health, Northumberland County Council. Westfield, Cramlington, Northumberland	
1931		* Whitson, Sir Thomas Barnby, D.L., LL.D., C.A., Lord Provost of the City of Edinburgh (1929–32), and President, Society of Accountants in Edinburgh (1937–38). 27 Eglinton Crescent, Edinburgh 12	
1911		Whittaker, Charles Richard, F.R.C.S.E., F.S.A.Scot., Lynwood, Hatton Place, Edinburgh 9	
1912	C. V. J. B-P.	Whittaker, Edmund Taylor, M.A., Hon. Sc.D. (Dubl.), Hon. LL.D. (St Andrews and California), Hon. D.Sc. (Nat. Univ. Ireland), F.R.S., Foreign Member of the R. Accademia dei Lincei, Rome, Member of the Pontifical Academy of Sciences, Corresponding Member of the R. Society of Naples, Professor of Mathematics, University of Edinburgh (Drummond Street) (President). 48 George Square, Edinburgh 8	1912–15, 1922–25. Sec. 1916–22. V-P 1925–28, 1937–39. P. 1939–
1928	C.	* Whittaker, John Macnaghten, M.A. (Edin.), M.A. (Cantab.), D.Sc., Professor of Pure Mathematics, University of Liverpool	
1936		* Whyte, Andrew, A.C.A., F.R.G.S., Chartered Accountant. The Knoll, 1 Linden Avenue, Darlington	
1918		* Whyte, Rev. Charles, M.A., LL.D., F.R.A.S., U.F. Church Manse, Kingswells, Aberdeen	
1935		* Whyte, Sir William Edward, Kt., O.B.E., J.P., Solicitor. Balgay, Uddingston	
1934		* Whyte, William, Cashier and General Manager of the Royal Bank of Scotland. Baberton House, Juniper Green, Edinburgh	
1929	C.	* Wiesner, Bertold Paul, D.Sc., formerly Macaulay Lecturer, Institute of Animal Genetics, University of Edinburgh. 37 Great Cumberland Place, London, W. 1	
1918		* Wight, John Thomas, M.I.Mech.E., M.I.Mar.E., Joint Managing Director, Messrs MacTaggart, Scott & Co., Ltd., Loanhead. Calderwood Villa, Lasswade	
1934		* Wightman, William Persehouse Delisle, Ph.D., M.Sc. (Lond.), Science Master, Edinburgh Academy. 36 Coltbridge Terrace, Edinburgh 12	1939–

Date of Election			Service on Council, etc.
1926	C. N.	* Williams, Samuel, Ph.D., Lecturer in Plant Morphology, University of Glasgow. 27 Lindsay Place, Kelvindale, Glasgow	
1908		Williamson, Henry Charles, M.A., D.Sc., formerly Naturalist to the Fishery Board for Scotland. 13 Windsor Street, Dundee	
1928	C.	* Williamson, John, M.A. (Edin.), Ph.D. (Chicago), Associate Professor of Mathematics, Johns Hopkins University, Baltimore, U.S.A.	
1910	C.	Williamson, William, F.L.S., 47 St Alban's Road, Edinburgh 9. *Temporary Address*, Sunnythwaite, West Linton	
1927	C.	* Williamson, William Turner Horace, B.Sc. (Aberd.), Ph.D. (Edin.), Chief Chemist, Egyptian Ministry of Agriculture, Cotton Research Board, Giza, Egypt. *Temporary Address*, 18 Northburn Avenue, Aberdeen	
1911		Wilson, Andrew, O.B.E., D.L., M.Inst.C.E. 51 Queen Street, Edinburgh 2	
1902	V. J.	† Wilson, Charles Thomson Rees, C.H., M.A., LL.D., D.Sc., F.R.S., Nobel Prize, Physics, 1927, Emeritus Professor of Natural Philosophy, University of Cambridge. 196 Grange Loan, Edinburgh 9	1937–
1922		* Wilson, John, F.R.I.B.A., Chief Architect, Scottish Department of Health. 20 Lomond Road, Edinburgh 5	
1920	C.	* Wilson, Malcolm, D.Sc. (Lond.), A.R.C.S., F.L.S., Reader in Mycology and Bacteriology, University of Edinburgh (Royal Botanic Garden). Brent Knoll, Kinnear Road, Edinburgh 4	1931–34.
1938		* Wilson, Robert, Master Printer, Partner of Messrs H. and J. Pillans & Wilson, Edinburgh. 13 Corrennie Drive, Edinburgh 10	
1924		* Wilson, William, M.A., LL.B., Advocate, Regius Professor of Public Law, University of Edinburgh (South Bridge). 38 Moray Place, Edinburgh 3	
1895		Wilson-Barker, Sir David, Kt., R.D., R.N.R., F.R.G.S., formerly Captain-Superintendent, Thames Nautical Training College, H.M.S. "Worcester." 12 Bolan Street, London, S.W. 11	
1934		* Winstanley, Arthur, M.B.E., D.Sc. Eng. (Lond.), M.I.Min.E., Mining Engineer to Safety in Mines Research Board. 18 St John's Road, Edinburgh 12	
1939		* Wishart, George Macfeat, B.Sc., M.D. (Glas.), Professor of Physiological Chemistry, University of Glasgow. 5 Hillhead Street, Glasgow, W. 2	
1931	C.	* Wishart, John, M.A., B.Sc. (Edin.), M.A. (Cantab.), D.Sc. (Lond.), Reader in Statistics, University of Cambridge. Croft Lodge, Barton Road, Cambridge	
1922	C. B.	* Wordie, James Mann, M.A. (Cantab.), B.Sc. (Glas.), St John's College, Cambridge	
1937		* Wright, Edward Maitland, B.A. (Lond.), M.A., D.Phil. (Oxon.), Professor of Mathematics, University of Aberdeen. 52 College Bounds, Aberdeen	
1933	C.	* Wright, James, F.G.S., Balado, 212 Colinton Road, Edinburgh 11	
1911	C.	Wrigley, Ruric Whitehead, M.A. (Cantab.), Assistant Astronomer, Royal Observatory, Edinburgh	
1938		* Wyburn, George McCreath, M.B., Ch.B. (Glas.), F.R.F.S.P.G., Lecturer in Anatomy, University of Glasgow	
1939		* Yarrow, Sir Harold Edgar, Bart., C.B.E., Chairman of Yarrow & Co., Ltd., Shipbuilders, etc. Craigend Castle, Milngavie, Stirlingshire.	
1937	C.	* Young, Andrew White, M.A., B.Sc., LL.B. (Edin.), W.S. 15 Great Stuart Street, Edinburgh 3	
1882		Young, Frank W., C.B.E., F.C.S., H.M. Inspector of Schools (Emeritus). Pancra, Shortheath, Farnham, Surrey	
1904		Young, Robert B., M.A., D.Sc., F.G.S., formerly Professor of Geology, University of the Witwatersrand (South African School of Mines and Technology), Johannesburg, Transvaal. St Patrick's Road, Houghton, Johannesburg, South Africa	

Number of Fellows, 765.

HONORARY FELLOWS OF THE SOCIETY.

(*At 23rd October* 1939.)

1920 HIS ROYAL HIGHNESS THE DUKE OF WINDSOR, K.G.

FOREIGNERS (LIMITED TO FORTY-FOUR BY LAW I).

Elected

1936 Leo Hendrik Baekeland, Professor (Honorary) of Chemical Engineering, Columbia University, New York. Bakelite Corporation, 247 Park Avenue, New York City.

1916 Charles Eugène Barrois, formerly Professor of Geology and Mineralogy, Université, Lille, France. 41, Rue Pascal, Lille. (*Died 8th December* 1939.)

1923 Henri Bergson, Honorary Professor, College of France, Paris.

1930 Vilhelm Frimann Koren Bjerknes, Professor of Physics, University of Oslo.

1937 Marston Taylor Bogert, Emeritus Professor of Organic Chemistry, Columbia University, N.Y.

1927 Niels Bohr, Nobel Laureate, Physics, 1922, Professor of Physics, University of Copenhagen.

1927 Jules Bordet, Nobel Laureate, Medicine, 1919, Professor of Bacteriology, University of Brussels.

1933 Filippo Bottazzi, Professor of Experimental Physiology, Royal Institute of Physiology, S. Andrea delle Dame, 21, Naples.

1923 Marcellin Boule, Director of the Institute of Human Palæontology, 1, Rue René-Panhard, Paris, XIII^{e}.

1905 Waldemar Christofer Brögger, Professor of Mineralogy and Geology, K. Frederiks Universitet, Oslo.

1916 Douglas Houghton Campbell, Emeritus Professor of Botany, Stanford University, California.

1930 Walter Bradford Cannon, Professor of Physiology, Harvard Medical School, Boston, Mass.

1930 Maurice Caullery, Professor of Zoology in the University of Paris. Laboratoire d'Évolution des Êtres organisés, 105 Bould. Raspail, Paris, VI^{e}.

1933 Edwin Grant Conklin, formerly Professor of Biology, Princeton University, N.J.

1939 Harvey (Williams) Cushing, Emeritus Professor of Neurology, Yale School of Medicine, New Haven, Conn., U.S.A. (*Died 7th October* 1939.)

1921 Reginald Aldworth Daly, Professor of Geology, Harvard University, Cambridge, Mass.

1927 Albert Einstein, Nobel Laureate, Physics, 1921, Princeton University, N.J.

1938 Federigo Enriques, Professor of Mathematics, Royal University, Rome.

1934 Björn Helland-Hansen, Geophysical Institute, Bergen.

1921 Johan Hjort, Professor of Marine Biology, University, Oslo.

1923 Arnold Frederik Holleman, Emeritus Professor of Organic Chemistry, University, Amsterdam. Boekensteijn Parkweg 7, Bloemendaal.

1934 Bernardo Houssay, Professor of Physiology, National University of Buenos Aires.

1937 C. U. Ariëns Kappers, Director of the Central Institute of Brain Research, Amsterdam, and Professor of Comparative Neurology, University, Amsterdam.

1933 Nikolaj Konstantinovic Koltzoff, formerly Professor of Zoology, State University, Moscow; Director of the Research Institute of Experimental Biology. Moscow 64, Voronzovo Pole 6.

1923 Tullio Levi-Civita, Professor of Mathematics, Regia Università, Rome.

1934 Frank Rattray Lillie, Emeritus Professor of Zoology and Embryology, University of Chicago, and President, National Academy of Sciences, Washington, D.C.

1939 Otto Loewi, lately Professor of Pharmacology, University of Graz, Austria. C/o National Institute for Medical Research, Mount Vernon, London, N.W. 3.

1936 Maurice Lugeon, Professor of Geology, University, Lausanne.

1939 Bernard Lyot, For. Assoc. Roy. Astron. Soc., l'Observatoire, Meudon (S. et O.), France.

1927 Hans Horst Meyer, Emeritus Professor of Pharmacology, University of Vienna.

1934 Thomas Hunt Morgan, Nobel Laureate, Medicine, 1933, Professor of Biology, California Institute of Technology, Pasadena.

1933 Albrecht Penck, Geheimrat, Emeritus Professor of Geography, Friedrich-Wilhelms-Universität, Berlin. Meierotto-Strasse 5^{11}, Berlin, W. 15.

1920 Charles Émile Picard, Perpetual Secretary, Academy of Sciences, Paris.

1937 Max Planck, Nobel Laureate, Physics, 1918, Professor Ordinarius Emeritus of Theoretical Physics, Director of the Institute for Theoretical Physics, University of Berlin.

Elected

1938 Henry Norris Russell, Chairman of the Department of Astronomy and Director of the Observatory, Princeton University, U.S.A.

1934 Paul Sabatier, Nobel Laureate, Chemistry, 1912, Professor of Chemistry, University of Toulouse.

1936 George Sarton, Editor of *Isis* and *Osiris*, Harvard Library, 185, Cambridge, Mass.

1930 Erik Helge Oswald Stensiö, Professor of Palæontology and Historical Geology, Royal University of Upsala.

1936 George Linius Streeter, Director, Department of Embryology, Carnegie Institution of Washington, Corner Wolfe and Madison Streets, Baltimore. 3707 St Paul Street, Baltimore, Md.

1938 Karl Freiherr von Tubeuf, Professor of Botany, University of Munich.

1936 Nikolai Ivanovic Vavilov, Director of the Institute of Plant Industry, Academy of Sciences, Leningrad, U.S.S.R.

1913 Vito Volterra, formerly Professor of Mathematical Physics, Regia Università. Via in Lucina 17, Rome.

1927 Richard Willstätter, Nobel Laureate, Chemistry, 1915. Muralto-Locarno (Switzerland), Villa L'Gremitaggio.

1933 Pieter Zeeman, Nobel Laureate, Physics, 1902, Emeritus Professor of Physics, University, Amsterdam. Stadhouderskade 158, Amsterdam.

Total, 43.

BRITISH SUBJECTS (LIMITED TO TWENTY-TWO BY LAW I).

1937 John Logie Baird, Inventor of the Televisor. 3 Crescent Wood Road, Sydenham, London, S.E.

1936 Sir Charles Vernon Boys, Kt., A.R.S.M., LL.D., F.R.S., St Marybourne, Andover.

1927 Sir William Henry Bragg, O.M., K.B.E., M.A., Hon. Sc.D. (Cantab.), Hon. D.Sc., Hon. D.C.L. (Durh.), LL.D., President R.S., Nobel Laureate, Physics, 1915, Fullerian Professor of Chemistry, Royal Institution, London.

1937 William Thomas Calman, C.B., D.Sc., LL.D. (St Andrews), F.R.S., F.L.S., formerly Keeper of Zoology, British Museum. Willowbrae, Tayport, Fife.

1936 Sir Henry Hallett Dale, Kt., C.B.E., M.A., M.D., LL.D., Hon. D.Sc., F.R.S., Joint Nobel Laureate, Medicine, 1936, Director of the National Institute for Medical Research, Mount Vernon, London, N.W. 3.

1936 Frederick George Donnan, C.B.E., M.A., Ph.D., D.Sc., LL.D., F.R.S., formerly Professor of Chemistry, University of London, and Director of Chemical Laboratories, University College. 23 Woburn Square, London, W.C. 1.

1930 Sir Arthur Stanley Eddington, Kt., O.M., M.A., Hon. D.Sc., LL.D., F.R.S., Plumian Professor of Astronomy and Experimental Philosophy, University of Cambridge.

1927 Sir John Bretland Farmer, Kt., M.A., D.Sc., LL.D., F.R.S., Emeritus Professor of Botany, Imperial College of Science and Technology, London. St Leonard's, Weston Road, Bath.

1900 Andrew Russell Forsyth, M.A., Sc.D., LL.D., Hon. Math.D., F.R.S., Emeritus Professor of Mathematics, Imperial College of Science and Technology, London; formerly Sadleirian Professor of Pure Mathematics, University of Cambridge. Bailey's Hotel, London, S.W. 7.

1910 Sir James George Frazer, Kt., O.M., D.C.L., LL.D., Litt.D., F.B.A., F.R.S., Commandeur de la Légion d'Honneur. Trinity College, Cambridge.

1927 Sir Frederick Gowland Hopkins, Kt., O.M., M.A., M.B., Hon. D.Sc., D.C.L. (Durh.), LL.D., Past President R.S., Joint Nobel Laureate, Medicine, 1929, Sir William Dunn Professor of Biochemistry, University of Cambridge. 71 Grange Road, Cambridge.

1930 Sir Arthur Keith, Kt., M.D., LL.D., D.Sc., F.R.S., Master, Buckston Browne Research Farm, Downe, Farnborough, Kent.

1910 Sir Joseph Larmor, Kt., M.A., D.Sc., LL.D., D.C.L. (Durh.), F.R.S., formerly Lucasian Professor of Mathematics, University of Cambridge. St John's College, Cambridge.

1938 Sir Thomas Lewis, Kt., C.B.E., D.Sc., M.D., F.R.C.P. (Lond.), Hon. F.R.C.P.E., LL.D., F.R.S., Consulting Physician, Ministry of Pensions, Physician-in-Charge of Department of Clinical Research, University College Hospital, London.

1933 Sir George Macdonald, K.C.B., D.Litt., Litt.D., LL.D., F.B.A., Hon. R.S.A., formerly Secretary, Scottish Education Department. 17 Learmonth Gardens, Edinburgh 4.

1934 Sir Edward Bagnall Poulton, Kt., M.A., D.Sc., LL.D., Hon. D.Sc., F.R.S., formerly Hope Professor of Zoology, University of Oxford. Wykeham House, Banbury Road, Oxford.

1930 Sir Robert Robinson, Kt., M.A., D.Sc., Hon. D.Sc., LL.D., F.R.S., Waynflete Professor of Chemistry, University of Oxford.

Elected

1933 Sir William Napier Shaw, Kt., Sc.D. (Cantab.), LL.D., Hon. Sc.D., Hon. D.Sc., F.R.S., formerly Director Meteorological Office. 171 Old Brompton Road, London, S.W. 5.

1908 Sir Charles Scott Sherrington, O.M., G.B.E., M.A., D.Sc., M.D., LL.D., Past President R.S., Joint Laureate, Nobel Prize, Medicine, 1932, formerly Waynflete Professor of Physiology, University of Oxford. Broomside, Valley Road, Ipswich.

1938 Geoffrey Ingram Taylor, M.A., Hon. D.Sc., D.C.L. (Brit. Columbia), F.R.S., Yarrow Research Professor of the Royal Society, Fellow of Trinity College, Cambridge.

1905 Sir Joseph John Thomson, Kt., O.M., D.Sc., LL.D., Past President R.S., Nobel Laureate, Physics, 1906, formerly Cavendish Professor of Experimental Physics, now Professor of Physics, University of Cambridge, Master of Trinity College, Cambridge.

1934 William Whitehead Watts, Sc.D., M.Sc., LL.D., F.R.S., Emeritus Professor of Geology, Imperial College of Science and Technology, London. Hillside, 39 Langley Park, Sutton, Surrey.

Total, 22.

CHANGES IN FELLOWSHIP DURING SESSION 1938–1939.

FELLOWS OF THE SOCIETY ELECTED.

ARTHUR BARTON PILGRIM AMIES
WILLIAM GILLIES ANNAN
THE REV. JAMES HOUSTON BAXTER
CECIL ARNOLD BEEVERS
AMULYARATAN CHAKRAVARTI
JAMES DAVIDSON
FRANCIS DAVIES
VICTOR AMBROSE EYLES
IAN FRASER
JOHN GALLOWAY GALLOWAY
BIRENDRA NATH GHOSH
WILLIAM MICHAEL HERBERT GREAVES
WILLIAM ROBERT HALL
WALTER FEARN HARPER
NORMAN MILLER JOHNSON
STENARD ERNEST ANDREW LANDALE
CYRIL EDWARD LUCAS
JAMES WRIGHT MACFARLANE
JOHN IAN GRAHAM MACGREGOR
JAMES MACALISTER MACKINTOSH
JOHN WILLIAM McNEE
WILLIAM MAIR
DONALD CAPELL MATHESON
JOHN BARRÉ DE WINTON MOLONY
ALASTAIR CAMPBELL MURRAY
WILLIAM DOUGLAS OLIPHANT
SIR ARTHUR OLVER
WILLIAM ANGUS SINCLAIR
BERNARD HALLEY STEWART
GEORGE MACFEAT WISHART
SIR HAROLD EDGAR YARROW, BART.

HONORARY FELLOWS ELECTED.

FOREIGN.

HARVEY (WILLIAMS) CUSHING
OTTO LOEWI
BERNARD LYOT

FELLOWS DECEASED.

GEORGE BARGER
SIR JAMES BARR
SIR ARCHIBALD DENNY. (*Died May* 1936)
SIR FRANK WATSON DYSON
SIR FREDERICK T. G. HOBDAY
JAMES GALL INGLIS
COL. HENRY HALCRO JOHNSTON
WALTER JOHN MABBOTT
TARAK NATH MAJUMDAR
JOHN MURDOCH MURRAY
VERY REV. WILLIAM PATERSON PATERSON
SIR ROBERT WILLIAM PHILIP
THOMAS STEPHENSON
DAVID W. SUTHERLAND
ARTHUR LOGAN TURNER
ALEXANDER G. WALLACE
ROBERT WALLACE
SIR ROBERT PATRICK WRIGHT

FOREIGN HONORARY FELLOWS DECEASED.

HARVEY (WILLIAMS) CUSHING
EDMUND BEECHER WILSON

RESIGNED.

CORBET PAGE STEWART

FELLOW REMOVED FROM ROLL.

OWEN FIENNES TEMPLE ROBERTS

LAWS OF THE SOCIETY.

Adopted July 3, 1916; *amended December* 18, 1916.

LAW I, amended February 5, 1934.	LAW IX, amended May 3, 1920.
,, VI, ,, ,, 7, 1921.	,, XIII, ,, May 3, 1920.
,, ,, ,, July 2, 1928.	,, XIX, ,, June 16, 1924.
,, VIII, ,, May 3, 1920.	

I.

THE ROYAL SOCIETY OF EDINBURGH, which was instituted by Royal Charter in 1783 for the promotion of Science and Literature, shall consist of Ordinary Fellows (hereinafter to be termed Fellows) and Honorary Fellows. The number of Honorary Fellows shall not exceed sixty-six, of whom not more than twenty-two may be British subjects, and not more than forty-four subjects of Foreign States.

Fellows only shall be eligible to hold office or to vote at any Meeting of the Society.

ELECTION OF FELLOWS.

II.

Each Candidate for admission as a Fellow shall be proposed by at least four Fellows, two of whom must certify from personal knowledge. The Official Certificate shall specify the name, rank, profession, place of residence, and the qualifications of the Candidate. The Certificate shall be delivered to the General Secretary before the 30th of November, and, subject to the approval of the Council, shall be exhibited in the Society's House during the month of January following. All Certificates so exhibited shall be considered by the Council at its first meeting in February, and a list of the Candidates approved by the Council for election shall be issued to the Fellows not later than the 21st of February.

III.

The election of Fellows shall be by Ballot, and shall take place at the first Ordinary Meeting in March. Only Candidates approved by the Council shall be eligible for election. A Candidate shall be held not elected, unless he is supported by a majority of two-thirds of the Fellows present and voting.

IV.

On the day of election of Fellows two scrutineers, nominated by the President, shall examine the votes and hand their report to the President, who shall declare the result.

V.

Each Fellow, after his election, is expected to attend an Ordinary Meeting, and sign the Roll of Fellows, he having first made the payments required by Law VI. He shall be introduced to the President, who shall address him in these words:—

> *In the name and by the authority of THE ROYAL SOCIETY OF EDINBURGH, I admit you a Fellow thereof.*

PAYMENTS BY FELLOWS.

VI.

Each Fellow shall, before he is admitted to the privileges of Fellowship, pay an admission fee of three guineas, and a subscription of three guineas for the year of election. He shall continue to pay a subscription of three guineas at the beginning of each session so long as he remains a Fellow.

Each Fellow who was elected subsequent to December 1916 and previous to December 1920 shall also pay a subscription of three guineas at the beginning of each session so long as he remains a Fellow.

Each Fellow who was elected previous to December 1916, and who has not completed his twenty-five annual payments, shall, at the beginning of each session, pay three guineas until his twenty-five annual payments are made. Each Fellow who has completed or shall complete his payments shall be invited to contribute one guinea per annum or to pay a single sum of ten guineas.

A Fellow may compound for the annual subscriptions by a single payment of fifty guineas, or on such other terms as the Council may from time to time fix.

VII.

A Fellow who, after application made by the Treasurer, fails to pay any contribution due by him, shall be reported to the Council, and, if the Council see fit, shall be declared no longer a Fellow. Notwithstanding such declaration all arrears of contributions shall remain exigible.

ELECTION OF HONORARY FELLOWS.

VIII.

Honorary Fellows shall be persons eminently distinguished in Science or Literature. They shall not be liable to contribute to the Society's Funds. Personages of the Blood Royal may be elected Honorary Fellows at any time on the nomination of the Council, and without regard to the limitation of numbers specified in Law I.

IX.

Honorary Fellows shall be proposed by the Council. The nominations shall be announced from the Chair at the First Ordinary Meeting after their selection. The names shall be printed in the circular for the last Ordinary Meeting of the Session, when the election shall be by Ballot, after the manner prescribed in Laws III and IV for the Election of Fellows.

EXPULSION OF FELLOWS.

X.

If, in the opinion of the Council, the conduct of any Fellow is injurious to the character or interests of the Society, the Council may, by registered letter, request him to resign. If he fail to do so within one month of such request, the Council shall call a Special Meeting of the Society to consider the matter. If a majority consisting of not less than two-thirds of the Fellows present and voting decide for expulsion, he shall be expelled by declaration from the Chair, his name shall be erased from the Roll, and he shall forfeit all right or claim in or to the property of the Society.

XI.

It shall be competent for the Council to remove any person from the Roll of Honorary Fellows if, in their opinion, his remaining on the Roll would be injurious to the character or interests of the Society. Reasonable notice of such proposal shall be given to each member of the Council, and, if possible, to the Honorary Fellow himself. Thereafter the decision on the question shall not be taken until the matter has been discussed at two Meetings of Council, separated by an interval of not less than fourteen days. A majority of two-thirds of the members present and voting shall be required for such removal.

MEETINGS OF THE SOCIETY.

XII.

A Statutory Meeting for the election of Council and Office-Bearers, for the presentation of the Annual Reports, and for such other business as may be arranged by the Council, shall be held on the fourth Monday of October. Each Session of the Society shall begin at the date of the Statutory Meeting.

XIII.

Meetings for reading and discussing communications and for general business, herein termed Ordinary Meetings, shall be held, when convenient, on the first and third Mondays of each month from November to July inclusive, with the exception that in January the meetings shall be held on the second and fourth Mondays.

XIV.

A Special Meeting of the Society may be called at any time by direction of the Council, or on a requisition to the Council signed by not fewer than six Fellows. The date and hour of such Meeting shall be determined by the Council, who shall give not less than seven days' notice of such Meeting. The notice shall state the purpose for which the Special Meeting is summoned; no other business shall be transacted.

PUBLICATION OF PAPERS.

XV.

The Society shall publish Transactions and Proceedings. The consideration of the acceptance, reading, and publication of papers is vested in the Council, whose decision shall be final. Acceptance for reading shall not necessarily imply acceptance for publication.

DISTRIBUTION OF PUBLICATIONS.

XVI.

Fellows who are not in arrear with their Annual Subscriptions and all Honorary Fellows shall be entitled gratis to copies of the Parts of the Transactions and the Proceedings published subsequently to their admission.

Copies of the Parts of the Proceedings shall be distributed by post or otherwise, as soon as may be convenient after publication; copies of the Transactions or Parts thereof shall be obtainable upon application, either personally or by an authorised agent, to the Librarian, provided the application is made within five years after the date of publication.

CONSTITUTION OF COUNCIL.

XVII.

The Council shall consist of a President, six Vice-Presidents, a Treasurer, a General Secretary, two Secretaries to the Ordinary Meetings (the one representing the Biological group and the other the Physical group of Sciences),* a Curator of the Library and Museum, and twelve ordinary members of Council.

ELECTION OF COUNCIL.

XVIII.

The election of the Council and Office-Bearers for the ensuing Session shall be held at the Statutory Meeting on the fourth Monday of October. The list of the names recommended by the Council shall be issued to the Fellows not less than one week before the Meeting. The election shall be by Ballot, and shall be determined by a majority of the Fellows present and voting. Scrutineers shall be nominated as in Law IV.

XIX.

The President may hold office for a period not exceeding five consecutive years; the Vice-Presidents, not exceeding three; the Secretaries to the Ordinary Meetings, not exceeding five; the General Secretary, the Treasurer, and the Curator of the Library and Museum, not exceeding ten; and ordinary members of Council, not exceeding three consecutive years; provided that the Treasurer may be re-elected for more than ten successive years in cases where the Council declares to the Society that an emergency exists.

XX.

In the event of a vacancy arising in the Council or in any of the Offices enumerated in Law XVII, the Council shall proceed, as soon as con-

* The Biological group includes Anatomy, Anthropology, Botany, Geology, Pathology, Physiology, Zoology; the Physical group includes Astronomy, Chemistry, Mathematics, Metallurgy, Meteorology, Physics.

venient, to elect a Fellow to fill such vacancy for the period up to the next Statutory Meeting.

POWERS OF THE COUNCIL.

XXI.

The Council shall have the following powers:—(1) To manage all business concerning the affairs of the Society. (2) To decide what papers shall be accepted for communication to the Society, and what papers shall be printed in whole or in part in the Transactions and Proceedings. (3) To appoint Committees. (4) To appoint employees and determine their remuneration. (5) To award the various prizes vested in the Society, in accordance with the terms of the respective deeds of gift, provided that no member of the existing Council shall be eligible for any such award. (6) To make from time to time Standing Orders for the regulation of the affairs of the Society. (7) To control the investment or expenditure of the Funds of the Society.

At Meetings of the Council the President or Chairman shall have a casting as well as a deliberative vote.

DUTIES OF PRESIDENT AND VICE-PRESIDENTS.

XXII.

The President shall take the Chair at Meetings of Council and of the Fellows. It shall be his duty to see that the business is conducted in accordance with the Charter and Laws of the Society. When unable to be present at any Meetings or attend to current business, he shall give notice to the General Secretary, in order that his place may be supplied. In the absence of the President his duties shall be discharged by one of the Vice-Presidents.

DUTIES OF THE TREASURER.

XXIII.

The Treasurer shall receive the monies due to the Society and shall make payments authorised by the Council. He shall lay before the Council a list of arrears in accordance with Rule VII. He shall keep accounts of all receipts and payments, and at the Statutory Meeting shall present the accounts for the preceding Session, balanced to the 30th of September, and audited by a professional accountant appointed annually by the Society.

DUTIES OF THE GENERAL SECRETARY.

XXIV.

The General Secretary shall be responsible to the Council for the conduct of the Society's correspondence, publications, and all other business except that which relates to finance. He shall keep Minutes of the Statutory and Special Meetings of the Society and Minutes of the Meetings of Council. He shall superintend, with the aid of the Assistant Secretary, the publication of the Transactions and Proceedings. He shall supervise the employees in the discharge of their duties.

DUTIES OF SECRETARIES TO ORDINARY MEETINGS.

XXV.

The Secretaries to Ordinary Meetings shall keep Minutes of the Ordinary Meetings. They shall assist the General Secretary, when necessary, in superintending the publication of the Transactions and Proceedings. In his absence, one of them shall perform his duties.

DUTIES OF CURATOR OF LIBRARY AND MUSEUM.

XXVI.

The Curator of the Library and Museum shall have charge of the Books, Manuscripts, Maps, and other articles belonging to the Society. He shall keep the Card Catalogue up to date. He shall purchase Books sanctioned by the Council.

ASSISTANT SECRETARY AND LIBRARIAN.

XXVII.

The Council shall appoint an Assistant Secretary and Librarian, who shall hold office during the pleasure of the Council. He shall give all his time, during prescribed hours, to the work of the Society, and shall be paid according to the determination of the Council. When necessary he shall act under the Treasurer in receiving subscriptions, giving out receipts, and paying employees.

ALTERATION OF LAWS.

XXVIII.

Any proposed alteration in the Laws shall be considered by the Council, due notice having been given to each member of Council. Such alteration, if approved by the Council, shall be proposed from the Chair at the next Ordinary Meeting of the Society, and, in accordance with the Charter, shall be considered and voted upon at a Meeting held at least one month after that at which the motion for alteration shall have been proposed.

Additions to the Library—Presentations, etc.—1938-1939.

(*Presented by Mr William Williamson, F.R.S.E., F.L.S.*)

Bonanni, Philipp. . . . cum Micrographia curiosa sive rerum minutissimarum Observationibus, . . . His accesserunt aliquot Animalium Testaceorum icones. (Lacking title page.) 4to. Romæ, 1691.

The Herball or Generall Historie of Plantes. Gathered by John Gerarde of London, Master in Chirvrgerie. Very much Enlarged and Amended by Thomas Johnson, Citizen and Apothecarye of London. Folio. London, 1636.

Festschrift des Vereins für Naturkunde zu Cassel zur Feier seines fünfzigjährigen Bestehens. 8vo. Cassel, 1886.

Gage, Simon Henry, and Gage, Henry Phelps. Optic Projection. La. 8vo. Ithaca, N.Y., 1914.

Leighton, Gerald. Botulism and Food Preservation (The Loch Maree Tragedy). 8vo. London, 1923.

Manning, Henry P. The Fourth Dimension Simply Explained. 8vo. London, 1921.

Regan, C. Tate. The Freshwater Fishes of the British Isles. 8vo. London, 1911.

Regnard, Paul. Recherches expérimentales sur les Conditions physiques de la Vie dans les Eaux. La. 8vo. Paris, 1891.

Reiche, Fritz. The Quantum Theory. Translated by H. S. Hatfield and Henry L. Bose. 8vo. London, 1922.

Reinhardt, Ludwig. Der Mensch zur Eiszeit in Europa und seine Kulturentwicklung bis zum Ende der Steinzeit. 8vo. München, 1906.

Seeley, H. G. The Fresh-water Fishes of Europe, a History of their Genera, Species, Structure, Habits, and Distribution. La. 8vo. London, 1886.

Sommerfeld, Arnold. Atomic Structure and Spectral Lines. Translated from the 3rd German Edition by Henry L. Bose. 8vo. London, 1923.

Allen, Glover M. The Mammals of China and Mongolia. Part I. (Central Asiatic Expeditions of the American Museum of Natural History.—The Natural History of Central Asia, Vol. XI.) 4to. New York, 1938. (*Presented.*)

Avhandlingar i naturskyddsärenden. No. 1→. Published by K. Svenska Vetenskapsakademiens. La. 8vo. Stockholm, 1938. (*Exchange.*)

Bailey, E. B., Weir, J., and McCallien, W. J. Introduction to Geology: General Geology by E. B. Bailey; Palæontology by J. Weir; and Illustrations by W. J. McCallien. 8vo. London, 1939. (*Presented by the Authors.*)

Berzelius. Berzelius—Porträtt: Illustrerad Beskrivning av Arne Holmberg. La. 8vo. Uppsala, 1939. (*Presented.*)

Bibliography of References to the Literature on the Minor Elements and Their Relation to Plant and Animal Nutrition. Third Edition. Originally Compiled by L. G. Willis. 4to. No Town. No Date. [1939.] (*Presented by the Nitrate Corporation of Chile, Limited, London.*)

Boys, Sir Charles Vernon. Weeds, Weeds, Weeds. 2nd and Enlarged Ed. 8vo. London, 1938. (*Presented by the Author.*)

Bristowe, William Syer. The Comity of Spiders. Vol. I. (Ray Society Publication.) 8vo. London, 1939. (*Purchased.*)

British Association Report for 1938. 8vo. London, 1938. (*Presented by Dr James Watt.*)

British Chemical Abstracts: Section A: III. Physiological and Biochemical Abstracts. 1937→. 4to. London, 1937. (*Presented by the Library, Chemistry Department, University of Edinburgh.*)

Bulletin of the Fan Memorial Institute of Biology. (Zoology.) Vol. VII, No. 1→. 4to. Peiping, 1936. (*Exchange.*)

Bulletin of the Geological Survey of Great Britain. No. 1→. 8vo. London, 1939. (*Exchange.*)

Carnegie Institution of Washington:—

(*Exchange.*)

No. 437. The Inscriptions of Peten. By Sylvanus Griswold Morley. Vols. I–V. 4to. 1937.

No. 487. Contributions to Palæontology. Studies on Cenozoic Vertebrates of Western North America. By Robert W. Wilson and others. 8vo. 1938.

No. 488. Chapters in the History of Social Legislation in the United States to 1860. By Henry W. Farnam. 8vo. 1938.

No. 490. Leaf Xanthophylis. By Harold H. Strain. 8vo. 1938.

No. 492. The Environment and History of the Toroweap and Kaibab Formations of Northern Arizona and Southern Utah. By Edwin D. McKee. 8vo. 1938.

No. 497. Hibernation and Marmot Physiology. By Francis G. Benedict and Robert C. Lee. 8vo. 1938.

No. 498. Magnetic Observations of Sunspots, 1917–1924. Parts 1 and 2. By George E. Hale and Seth B. Nicholson. 4to. 1938.

No. 499. A Medical Survey of the Republic of Guatemala. By George G. Shattuck and others. 4to. 1938.

No. 500. Published Papers and Addresses of John Campbell Merriam. Vols. I–IV. 8vo. 1938.

No. 501. Cooperation in Research. By Staff Members and Research Associates, The Carnegie Institution of Washington. 8vo. 1938.

No. 502. Modern Maya Houses. A Study of their Archæological Significance. By Robert Wauchope. 4to. 1938.

No. 504. American Species of Crepis. Their Interrelationships and Distribution as Affected by Polyploidy and Apomixis. By E. B. Babcock and G. L. Stebbins, Jr. 8vo. 1938.

Supplementary Publications:—

No. 38. The Raw Materials of Evolution. By Theodosius Dobzhansky. 8vo. 1938.

No. 39. Utilitarian Aspects of Geophysics. By A. G. McNish. 8vo. 1938.

No. 40. Prolactin, a Product of the Anterior Pituitary, and the Part it Plays in Vital Processes. By Oscar Riddle. 8vo. 1938.

No. 41. The Forces which Govern the Atomic Nucleus. By M. A. Tuve. 8vo. 1938.

No. 42. Application of Science in Human Affairs. By John C. Merriam. 8vo. 1938.

No. 43. Climatic Cycles and Human Populations in the Great Plains. By Frederic E. Clements. 8vo. 1938.

No. 44. Influence of Science upon Appreciation of Nature. By John C. Merriam. 8vo. 1938.

No. 45. Some Aspects of Cooperative Research in History. By John C. Merriam. 8vo. 1938.

Elihu Root Lectures, No. 5. Cultural Contacts of Science. By Sir Richard Gregory, Bart. 8vo. 1938.

News Service Bulletin. School Edition. Vols. II–IV. 4to. 1930–38.

Caullery, Maurice. Les progrès récents de l'embryologie expérimentale. 8vo. Paris, 1939. (*Presented by the Author.*)

Chemical Products and The Chemical News. Vol. I, No. 1→. 4to. London, 1938. (*Presented.*)

Collignon, Maurice. Paléontologie de Madagascar. XX.—Les Céphalopodes du Trias Inférieur; et XXII.—Les Ammonites Pyriteuses de l'Aptien d'Antanatanamirafy. (Extraits des Annales de Paléontologie.) 4to. Paris, 1933–34 et 1937. (*Exchange with Author.*)

IX Congreso Cientifico General Chileno. Celebrado en la ciudad de Valparaiso del 24 al 27 de Septiembre de 1936. Tomo I y II. 8vo. Santiago de Chile, 1938. (*Presented.*)

Contributed Technical Papers: National Geographic Society. Solar Eclipse Series. No. 1: National Geographic Society—U.S. Navy Solar Eclipse Expedition of 1937 to Canton Island. 8vo. Washington, 1939. (*Presented.*)

De Bary, August. Geschichte der Dr Senckenbergischen Stiftung, 1763–1938. 8vo. Frankfurt am Main, 1938. (*Presented by the Frankfurter Medizinische Gesellschaft.*)

Deutsches Entomologisches Institut:—

(*Exchange.*)

Arbeiten über physiologische und angewandte Entomologie aus Berlin-Dahlem. Vol. 1, Nr. 1→. 8vo. Berlin, 1934.

Arbeiten über morphologische und taxonomische Entomologie aus Berlin-Dahlem. Vol. 1, Nr. 1→. 8vo. Berlin, 1934.

Entomologische Beihefte aus Berlin-Dahlem. Vol. 1→. 8vo. Berlin, 1934.

Duke University:—

(*Exchange.*)

Duke Mathematical Journal. Vol. 4, No. 3→. 8vo. Durham, N.C., 1938.
Ecological Monographs. Vol. 8, No. 3→. 8vo. Durham, N.C., 1938.

The Early Maps of Scotland with an Account of the Ordnance Survey. By a Committee of the Royal Scottish Geographical Society. 2nd Ed. Revised. 8vo. Edinburgh, 1936. (*Presented by Mr J. D. Gardner, F.R.S.E.*)

Early Science in Oxford. By R. T. Gunther. Vol. XII: Dr Plot and the Correspondence of the Philosophical Society of Oxford. 8vo. Oxford, 1939. (*Purchased.*)

Enzyklopädie der mathematischen Wissenschaften. Vol. I: Algebra und Zahlentheorie. Teil 1, Heft 2 (2. Auflage). 8vo. Leipzig and Berlin, 1939. (*Purchased.*)

Fauna of British India:—

(*Presented.*)

Baylis, H. A. Nematoda. Vol. II (Filarioidea, Dioctophymoidea, and Trichinelloidea). 8vo. London, 1939.

Bhatia, B. L. Protozoa: Sporozoa. 8vo. London, 1938.

Cameron, Malcolm. Coleoptera: Staphylinidæ. Vol. IV, Parts 1 and 2. 8vo. London, 1939.

Pocock, R. I. Mammalia. Vol. I: Primates and Carnivora (in part), Families Felidæ and Viverridæ. 8vo. London, 1939.

Talbot, G. Butterflies. Vol. I. 8vo. London, 1939.

Feestnummer uitgegeven naar aanleiding van het Honderdjarig Bestaan van het Genootschap: 1838 mei 1938. (Bijdragen tot de Dierkunde: 27e Aflevering.) La. 8vo. Leiden, 1939. (*Exchange.*)

Fitzmaurice, R. Principles of Modern Building. Vol. I: Walls, Partitions, and Chimneys. 8vo. London, 1938. (*Presented by the Department of Scientific and Industrial Research, London.*)

Fowler, Henry W. The Fishes of the George Vanderbilt South Pacific Expedition, 1937. (Monographs: Academy of Natural Sciences of Philadelphia. No. 2.) 4to. Philadelphia, 1938. (*Exchange.*)

General Electric Review. Vol. XLII, No. 1→. 4to. New York, 1939. (*Purchased.*)

Geophysical Abstracts: United States Department of the Interior, Geological Survey Bulletin. No. 87→ (July–December 1936), contained within Bulletin 887, and later numbers as issued. 8vo. Washington, 1937. (*Exchange.*)

Grierson, Sir Herbert. Sir Walter Scott, Bart. A New Life supplementary to, and corrective of, Lockhart's Biography. 8vo. London, 1938. (*Purchased.*)

Gurney, Robert. Bibliography of the Larvæ of Decapod Crustacea. (Ray Society Publication.) 8vo. London, 1939. (*Purchased.*)

Hand-List of Short Titles of Current Periodicals in the Science Library. Part I: Alphabetical. Fifth Edition. 8vo. London, 1938. (*Presented.*)

Harrap's Standard French and English Dictionary. Edited by J. E. Mansion. Part Two: English-French. 4to. London, 1939. (*Purchased.*)

Hurst, H. E., and Phillips, P. The Nile Basin. Supplement to Vol. II. Measured Discharges of the Nile and its Tributaries from 1928 to 1932. (Physical Department Paper No. 33.) 4to. Cairo, 1938. (*Exchange.*)

—— The Nile Basin. Vol. V. The Hydrology of the Lake Plateau and Bahr El Jebel. (Physical Department Paper No. 35.) 8vo. Cairo, 1938. (*Exchange.*)

Industrial Research Laboratories of the United States including Consulting Research Laboratories. 6th Ed., 1938. (Bulletin, National Research Council, No. 102.) 8vo. Washington, 1938. (*Exchange.*)

W. & A. K. Johnston's Gazetteer of Scotland. 8vo. Edinburgh and London, 1937. (*Purchased.*)

Koltzoff, N. K. Les Molécules héréditaires. 8vo. Paris, 1939. (*Presented.*)

Lexicon of Geologic Names of the United States (including Alaska). Compiled by M. Grace Wilmarth. Parts 1, 2. (Bulletin 896: Geological Survey of the United States.) 8vo. Washington, 1938. (*Exchange.*)

Lloydia. A Quarterly Journal of Biological Science. Vol. I, No. 1→. Published by the Lloyd Library of Natural History. 8vo. Cincinnati, 1938. (*Exchange.*)

Lord Kelvin. William Thomson, Baron Kelvin of Largs: 1824–1907. (An Obituary Notice from Proceedings of the Royal Society of London: Series A. Vol. LXXXI, Appendix, 1908.) By J. L. La. 8vo. London, 1908. (*Presented by Sir Joseph Larmor with the additional page* (p. xxxviii).)

Lorentz, H. A. Collected Papers. Vol. IX. 4to. The Hague, 1939. (*Purchased.*)

Murdoch, William. Murdoch Centenary Lecture, 6–9 June 1939. By E. F. Armstrong. Published by the Institution of Gas Engineers. 8vo. London, 1939. (*Presented.*)

New England Naturalist. No. 1→. (Formerly Bulletin of the New England Museum of Natural History.) 8vo. Boston, 1938. (*Exchange with the Boston Society of Natural History.*)

Notes and Records. No. 2, October 1938→. Published by the Royal Society of London. 8vo. London, 1938. (*Exchange.*)

Notices et Discours. Académie des Sciences. Tome I→ (1924–1936). La. 4to. Paris, 1937. (*Exchange.*)

Princes Street, Edinburgh. An Illustrated Account of its Origin and Development. [By C. A. Malcolm.] Issued by the Life Association of Scotland on the occasion of its Centenary, 1938. 8vo. (*Presented.*)

Publications of the Royal Observatory, Edinburgh. Vol. I, No. 1→. 4to. Edinburgh, 1939. (*Exchange.*)

Raman, Sir C. V. In commemoration of the Fiftieth Birthday of, and Ten Years of Research on the Raman Effect. (Proceedings of the Indian Academy of Sciences. Vol. VIII, Section A, No. 5, November 1938.) (*Exchange.*)

Revue des Archives Italiennes de Biologie. Tome I, Fasc. 2→. 8vo. Pisa, 1938. (*Formerly:* (Nouvelles) Archives Italiennes de Biologie.) (*Purchased.*)

Ricerche Astronomiche. Vol. I, No. 1→. (Published by Specola Astronomica Vaticana.) 8vo. Città del Vaticano, 1939. (*Exchange.*)

Seismological Bulletin: Government of India Meteorological Department. Jan.–Mar. 1938→. 4to. Delhi, 1938. (*Formerly:* Part D, Annual Summary of the India Weather Review.) (*Exchange.*)

Seventy-five Years: A History of the Buffalo Society of Natural Science, 1861–1936. 8vo. Buffalo, N.Y., 1938. (Bulletin: Vol. XVIII.) (*Exchange.*)

Strand, Embrik. Festschrift zum 60 Geburtstage von Professor Dr Embrik Strand. Vol. V. 8vo. Riga, 1939. (*Presented.*)

Total Solar Eclipse of June 19, 1936: Reports of Soviet Expeditions. I. Published by the Academy of Sciences of U.S.S.R. 8vo. Moscow and Leningrad, 1939. (*Exchange.*)

The Total Solar Eclipse of 1940 October 1. (Supplement to the Nautical Almanac for 1940.) Fol. London, 1938. (*Exchange.*)

Transactions: International Federation for Documentation. XIVth Conference, Oxford-London, 1938. Vols. I–III. Fol. The Hague. (*Purchased.*)

University of Queensland Papers. Vol. I, No. 1→. 4to. Brisbane, 1937. (*Exchange.*)

Veendorp, H., and Baas Becking, L. G. M. Hortus Academicus Lugduno Batavus, 1587–1937: the Development of the Gardens of Leyden University. 4to. Haarlem, 1938. (*Presented.*)

La Via Claudia Augusta Altinate. Published by Reale Istituto Veneto di Scienze, Lettere ed Arti. 4to. Venezia, 1938. (*Presented.*)

Volume jubilaire dédié à Maurice Caullery. (Travaux de la Station zoologique de Wimereux. Tome XIII.) 4to. Paris, 1938. (*Purchased.*)

The Works on the Geology of the Sea. (Transactions of the Institute of Marine Fisheries and Oceanography of the U.S.S.R. Vol. V.) 8vo. Moscow and Leningrad, 1938. (*Exchange.*)

Zoogeographia: International Review for Comparative and Casual Animal Geography. Vol. 1, No. 1→. 8vo. Jena, 1932. (*Purchased.*)

INDEX

Index of Papers published in the "Transactions" during Session 1938–39.

PRINTED IN GREAT BRITAIN BY NEILL AND CO., LTD., EDINBURGH.